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# Large deviation principle and inviscid shell models

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**Abstract**

A LDP is proved for the inviscid shell model of turbulence. As the viscosity coefficient $\nu$ converges to 0 and the noise intensity is multiplied by $\sqrt{\nu}$, we prove that some shell models of turbulence with a multiplicative stochastic perturbation driven by a $H$-valued Brownian motion satisfy a LDP in $\mathscr{C}([0,T],V)$ for the topology of uniform convergence on $[0,T]$, but where $V$ is endowed with a topology weaker than the natural one. The initial condition has to belong to $V$ and the proof is based on the weak convergence of a family of stochastic control equations. The rate function is described in terms of the solution to the inviscid equation.

**Key words:** Shell models of turbulence, viscosity coefficient and inviscid models, stochastic PDEs, large deviations.

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## 1 Introduction

Shell models, from E.B. Gledzer, K. Ohkitani, M. Yamada, are simplified Fourier systems with respect to the Navier-Stokes ones, where the interaction between different modes is preserved only between nearest neighbors. These are some of the most interesting examples of artificial models of fluid dynamics that capture some properties of turbulent fluids like power law decays of structure functions.

There is an extended literature on shell models. We refer to K. Ohkitani and M. Yamada [25], V. S. Lvov, E. Podivilov, A. Pomyalov, I. Procaccia and D. Vandembroucq [21], L. Biferale [3] and the references therein. However, these papers are mainly dedicated to the numerical approach and pertain to the finite dimensional case. In a recent work by P. Constantin, B. Levant and E. S. Titi [11], some results of regularity, attractors and inertial manifolds are proved for deterministic infinite dimensional shells models. In [12] these authors have proved some regularity results for the inviscid case. The infinite-dimensional stochastic version of shell models have been studied by D. Barbato, M. Barsanti, H. Bessaih and F. Flandoli in [1] in the case of an additive random perturbation. Well-posedeness and apriori estimates were obtained, as well as the existence of an invariant measure. Some balance laws have been investigated and preliminary results about the structure functions have been presented.

The more general formulation involving a multiplicative noise reads as follows

$$du(t)+[\nu Au(t)+B(u(t),u(t))]\,dt=\sigma(t,u(t))\,dW_t\,,\quad u(0)=\xi.$$

driven by a Hilbert space-valued Brownian motion $W$. It involves some similar bilinear operator $B$ with antisymmetric properties and some linear "second order" (Laplace) operator $A$ which is regularizing and multiplied by some non negative coefficient $\nu$ which stands for the viscosity in the usual hydro-dynamical models. The shell models are adimensional and the bilinear term is better behaved than that in the Navier Stokes equation. Existence, uniqueness and several properties were studied in [1] in the case on an additive noise and in [10] for a multiplicative noise in the "regular" case of a non-zero viscosity coefficient which was taken constant.

Several recent papers have studied a Large Deviation Principle (LDP) for the distribution of the solution to a hydro-dynamical stochastic evolution equation: S. Sritharan and P. Sundar [27] for the 2D Navier Stokes equation, J. Duan and A. Millet [16] for the Boussinesq model, where the Navier Stokes equation is coupled with a similar nonlinear equation describing the temperature evolution, U. Manna, S. Sritharan and P. Sundar [22] for shell models of turbulence, I. Chueshov and A. Millet [10] for a wide class of hydro-dynamical equations including the 2D Bénard magneto-hydro dynamical and 3D $\alpha$-Leray Navier Stokes models, A.Du, J. Duan and H. Gao [15] for two layer quasi-geostrophic flows modeled by coupled equations with a bi-Laplacian. All the above papers consider an equation with a given (fixed) positive viscosity coefficient and study exponential concentration to a deterministic model when the noise intensity is multiplied by a coefficient $\sqrt{\epsilon}$ which converges to 0. All these papers deal with a multiplicative noise and use the weak convergence approach of LDP, based on the Laplace principle, developed by P. Dupuis and R. Ellis in [17]. This approach has shown to be successful in several other infinite-dimensional cases (see e.g. [4], [5], [20]) and differ from that used to get LDP in finer topologies for quasi-linear SPDEs, such as [26], [9], [7], [8]. For hydro-dynamical models, the LDP was proven in the natural space of trajectories, that is $\mathscr{C}([0,T],H)\cap L^2([0,T],V)$, where roughly speaking, $H$ is $L^2$ and $V=Dom(A^{\frac{1}{2}})$ is the Sobolev space $H_1^2$ with proper periodicity or boundary conditions. The initial condition $\xi$ only belongs to $H$.

The aim of this paper is different. Indeed, the asymptotics we are interested in have a physical meaning, namely the viscosity coefficient $\nu$ converges to 0. Thus the limit equation, which corresponds to the inviscid case, is much more difficult to deal with, since the regularizing effect of the operator $A$ does not help anymore. Thus, in order to get existence, uniqueness and apriori estimates to the inviscid equation, we need to start from some more regular initial condition $\xi \in V$, to impose that $(B(u,u),Au) = 0$ for all $u$ regular enough (this identity would be true in the case on the 2D Navier Stokes equation under proper periodicity properties); note that this equation is satisfied in the GOY and Sabra shell models of turbulence under a suitable relation on the coefficients $a, b$ and $\mu$ stated below. Furthermore, some more conditions on the diffusion coefficient are required as well. The intensity of the noise has to be multiplied by $\sqrt{\nu}$ for the convergence to hold.

The technique is again that of the weak convergence. One proves that given a family $(h_\nu)$ of random elements of the RKHS of $W$ which converges weakly to $h$, the corresponding family of stochastic control equations, deduced from the original ones by shifting the noise by $\frac{h_\nu}{\sqrt{\nu}}$, converges in distribution to the limit inviscid equation where the Gaussian noise $W$ has been replaced by $h$. Some apriori control of the solution to such equations has to be proven uniformly in $\nu > 0$ for "small enough" $\nu$. Existence and uniqueness as well as apriori bounds have to be obtained for the inviscid limit equation. Some upper bounds of time increments have to be proven for the inviscid equation and the stochastic model with a small viscosity coefficient; they are similar to that in [16] and [10]. The LDP can be shown in $\mathscr{C}([0,T],V)$ for the topology of uniform convergence on $[0,T]$, but where $V$ is endowed with a weaker topology, namely that induced by the $H$ norm. More generally, under some slight extra assumption on the diffusion coefficient $\sigma$, the LDP is proved in $\mathscr{C}([0,T],V)$ where $V$ is endowed with the norm $\|\cdot\|_\alpha := |A^\alpha(\cdot)|_H$ for $0 \le \alpha \le \frac{1}{4}$. The natural case $\alpha = \frac{1}{2}$ is out of reach because the inviscid limit equation is much more irregular. Indeed, it is an abstract equivalent of the Euler equation. The case $\alpha = 0$ corresponds to $H$ and then no more condition on $\sigma$ is required. The case $\alpha = \frac{1}{4}$ is that of an interpolation space which plays a crucial role in the 2D Navier Stokes equation. Note that in the different context of a scalar equation, M. Mariani [23] has also proved a LDP for a stochastic PDE when a coefficient $\varepsilon$ in front of a deterministic operator converges to 0 and the intensity of the Gaussian noise is multiplied by $\sqrt{\varepsilon}$. However, the physical model and the technique used in [23] are completely different from ours.

The paper is organized as follows. Section 2 gives a precise description of the model and proves apriori bounds for the norms in $\mathscr{C}([0,T],H)$ and $L^2([0,T],V)$ of the stochastic control equations uniformly in the viscosity coefficient $\nu \in ]0,\nu_0]$ for small enough $\nu_0$. Section 3 is mainly devoted to prove existence, uniqueness of the solution to the deterministic inviscid equation with an external multiplicative impulse driven by an element of the RKHS of $W$, as well as apriori bounds of the solution in $\mathscr{C}([0,T],V)$ when the initial condition belong to $V$ and under reinforced assumptions on $\sigma$. Under these extra assumptions, we are able to improve the apriori estimates of the solution and establish them in $\mathscr{C}([0,T],V)$ and $L^2([0,T],Dom(A))$. Finally the weak convergence and compactness of the level sets of the rate function are proven in section 4; they imply the LDP in $\mathscr{C}([0,T],V)$ where $V$ is endowed with the weaker norm associated with $A^\alpha$ for any value of $\alpha$ with $0 \le \alpha \le \frac{1}{4}$.

The LDP for the 2D Navier Stokes equation as the viscosity coefficient converges to 0 will be studied in a forthcoming paper.

We will denote by $C$ a constant which may change from one line to the next, and $C(M)$ a constant depending on $M$.

# 2 Description of the model

## 2.1 GOY and Sabra shell models

Let $H$ be the set of all sequences $u=(u_1,u_2,\dots)$ of complex numbers such that $\sum_n |u_n|^2<\infty$. We consider $H$ as a *real* Hilbert space endowed with the inner product $(\cdot,\cdot)$ and the norm $|\cdot|$ of the form

$$(u,v)=\mathrm{Re}\sum_{n\ge 1} u_n v_n^*, \quad |u|^2=\sum_{n\ge 1}|u_n|^2, \tag{2.1}$$

where $v_n^*$ denotes the complex conjugate of $v_n$. Let $k_0>0$, $\mu>1$ and for every $n\ge 1$, set $k_n=k_0\,\mu^n$. Let $A: Dom(A)\subset H\to H$ be the non-bounded linear operator defined by

$$(Au)_n=k_n^2 u_n, \quad n=1,2,\dots, \qquad Dom(A)=\Big\{u\in H\,:\,\sum_{n\ge 1}k_n^4|u_n|^2<\infty\Big\}.$$

The operator $A$ is clearly self-adjoint, strictly positive definite since $(Au,u)\ge k_0^2|u|^2$ for $u\in Dom(A)$. For any $\alpha>0$, set

$$\mathcal{H}_\alpha=Dom(A^\alpha)=\{u\in H\,:\,\sum_{n\ge 1}k_n^{4\alpha}|u_n|^2<+\infty\},\ \|u\|_\alpha^2=\sum_{n\ge 1}k_n^{4\alpha}|u_n|^2 \text{ for } u\in\mathcal{H}_\alpha. \tag{2.2}$$

Let $\mathcal{H}_0=H$,

$$V:=Dom(A^{\frac12})=\Big\{u\in H\,:\,\sum_{n\ge 1}k_n^2|u_n|^2<+\infty\Big\};\ \text{also set}\ \mathcal{H}=\mathcal{H}_{\frac14},\ \|u\|_{\mathcal{H}}=\|u\|_{\frac14}.$$

Then $V$ (as each of the spaces $\mathcal{H}_\alpha$) is a Hilbert space for the scalar product $(u,v)_V=Re(\sum_n k_n^2\,u_n\,v_n^*)$, $u,v\in V$ and the associated norm is denoted by

$$\|u\|^2=\sum_{n\ge 1}k_n^2\,|u_n|^2. \tag{2.3}$$

The adjoint of $V$ with respect to the $H$ scalar product is $V'=\{(u_n)\in\mathbb{C}^{\mathbb{N}}\,:\,\sum_{n\ge 1}k_n^{-2}\,|u_n|^2<+\infty\}$ and $V\subset H\subset V'$ is a Gelfand triple. Let $\langle u\,,\,v\rangle=Re\left(\sum_{n\ge 1}u_n\,v_n^*\right)$ denote the duality between $u\in V$ and $v\in V'$. Clearly for $0\le\alpha<\beta$, $u\in\mathcal{H}^\beta$ and $v\in V$ we have

$$\|u\|_\alpha^2\le k_0^{4(\alpha-\beta)}\,\|u\|_\beta^2,\ \text{and}\ \ \|v\|_{\mathcal{H}}^2\le|v|\,\|v\|, \tag{2.4}$$

where the last inequality is proved by the Cauchy-Schwarz inequality.

Set $u_{-1}=u_0=0$, let $a,b$ be real numbers and $B: H\times V\to H$ (or $B: V\times H\to H$) denote the bilinear operator defined by

$$[B(u,v)]_n=-i\left(ak_{n+1}u_{n+1}^*v_{n+2}^*+bk_nu_{n-1}^*v_{n+1}^*-ak_{n-1}u_{n-1}^*v_{n-2}^*-bk_{n-1}u_{n-2}^*v_{n-1}^*\right) \tag{2.5}$$

for $n=1,2,\dots$ in the GOY shell-model (see, e.g., [25]) or

$$[B(u,v)]_n=-i\left(ak_{n+1}u_{n+1}^*\,v_{n+2}+bk_nu_{n-1}^*v_{n+1}+ak_{n-1}u_{n-1}v_{n-2}+bk_{n-1}u_{n-2}v_{n-1}\right), \tag{2.6}$$

in the Sabra shell model introduced in [21].

Note that $B$ can be extended as a bilinear operator from $H \times H$ to $V'$ and that there exists a constant $C > 0$ such that given $u, v \in H$ and $w \in V$ we have

$$|\langle B(u,v)\,,\,w\rangle| + |\big(B(u,w)\,,\,v\big)| + |\big(B(w,u)\,,\,v\big)| \le C\,|u|\,|v|\,\|w\|. \tag{2.7}$$

An easy computation proves that for $u, v \in H$ and $w \in V$ (resp. $v, w \in H$ and $u \in V$),

$$\langle B(u,v)\,,\,w\rangle = -\big(B(u,w)\,,\,v\big) \text{ (resp. } \big(B(u,v)\,,\,w\big) = -\big(B(u,w)\,,\,v\big)\text{ ).} \tag{2.8}$$

Furthermore, $B : V \times V \to V$ and $B : \mathscr{H} \times \mathscr{H} \to H$; indeed, for $u, v \in V$ (resp. $u, v \in \mathscr{H}$) we have

$$\begin{aligned} \|B(u,v)\|^2 &= \sum_{n\ge 1} k_n^2\,|B(u,v)_n|^2 \le C\,\|u\|^2 \sup_n k_n^2 |v_n|^2 \le C\,\|u\|^2\,\|v\|^2, \\ |B(u,v)| &\le C\,\|u\|_{\mathscr{H}}\,\|v\|_{\mathscr{H}}. \end{aligned} \tag{2.9}$$

For $u, v$ in either $H$, $\mathscr{H}$ or $V$, let $B(u) := B(u,u)$. The anti-symmetry property (2.8) implies that $|\langle B(u_1) - B(u_2)\,,\,u_1 - u_2\rangle_V| = |\langle B(u_1 - u_2), u_2\rangle_V|$ for $u_1, u_2 \in V$ and $|\langle B(u_1) - B(u_2)\,,\,u_1 - u_2\rangle| = |\langle B(u_1 - u_2), u_2\rangle|$ for $u_1 \in H$ and $u_2 \in V$. Hence there exist positive constants $\bar{C}_1$ and $\bar{C}_2$ such that

$$|\langle B(u_1) - B(u_2)\,,\,u_1 - u_2\rangle_V| \;\le\; \bar{C}_1\,\|u_1 - u_2\|^2\,\|u_2\|, \forall u_1, u_2 \in V, \tag{2.10}$$

$$|\langle B(u_1) - B(u_2)\,,\,u_1 - u_2\rangle| \;\le\; \bar{C}_2\,|u_1 - u_2|^2\,\|u_2\|, \forall u_1 \in H, \forall u_2 \in V. \tag{2.11}$$

Finally, since $B$ is bilinear, Cauchy-Schwarz's inequality yields for any $\alpha \in [0, \frac{1}{2}]$, $u, v \in V$:

$$\begin{aligned} \Big|\big(A^\alpha B(u) - A^\alpha B(v)\,,\,A^\alpha(u-v)\big)\Big| &\le \Big|\big(A^\alpha B(u-v,u) + A^\alpha B(v,u-v)\,,\,A^\alpha(u-v)\big)\Big| \\ &\le C\|u-v\|_\alpha^2\,(\|u\| + \|v\|). \end{aligned} \tag{2.12}$$

In the GOY shell model, $B$ is defined by (2.5); for any $u \in V$, $Au \in V'$ we have

$$\langle B(u,u), Au\rangle = Re\Big(-i\sum_{n\ge 1} u_n^*\,u_{n+1}^*\,u_{n+2}^*\mu^{3n+1}\Big)\,k_0^3(a + b\mu^2 - a\mu^4 - b\mu^4).$$

Since $\mu \neq 1$,

$$a(1+\mu^2) + b\mu^2 = 0 \quad \text{if and only if } \langle B(u,u)\,,\,Au\rangle = 0, \forall u \in V. \tag{2.13}$$

On the other hand, in the Sabra shell model, $B$ is defined by (2.6) and one has for $u \in V$,

$$\langle B(u,u)\,,\,Au\rangle = k_0^3 Re\Big(-i\sum_{n\ge 1}\mu^{3n+1}\,\Big[(a + b\,\mu^2)\,u_n^*\,u_{n+1}^*\,u_{n+2} + (a+b)\mu^4 u_n\,u_{n+1}\,u_{n+2}^*\Big]\Big).$$

Thus $(B(u,u),Au) = 0$ for every $u \in V$ if and only if $a + b\mu^2 = (a+b)\mu^4$ and again $\mu \neq 1$ shows that (2.13) holds true.

## 2.2 Stochastic driving force

Let $Q$ be a linear positive operator in the Hilbert space $H$ which is trace class, and hence compact. Let $H_0 = Q^{\frac{1}{2}}H$; then $H_0$ is a Hilbert space with the scalar product

$$(\phi,\psi)_0 = (Q^{-\frac{1}{2}}\phi, Q^{-\frac{1}{2}}\psi),\ \forall \phi,\psi \in H_0,$$

together with the induced norm $|\cdot|_0 = \sqrt{(\cdot,\cdot)_0}$. The embedding $i : H_0 \to H$ is Hilbert-Schmidt and hence compact, and moreover, $i\, i^* = Q$. Let $L_Q \equiv L_Q(H_0, H)$ be the space of linear operators $S : H_0 \mapsto H$ such that $SQ^{\frac{1}{2}}$ is a Hilbert-Schmidt operator from $H$ to $H$. The norm in the space $L_Q$ is defined by $|S|^2_{L_Q} = tr(SQS^*)$, where $S^*$ is the adjoint operator of $S$. The $L_Q$-norm can be also written in the form

$$|S|^2_{L_Q} = tr([SQ^{1/2}][SQ^{1/2}]^*) = \sum_{k\geq 1} |SQ^{1/2}\psi_k|^2 = \sum_{k\geq 1} |[SQ^{1/2}]^*\psi_k|^2 \tag{2.14}$$

for any orthonormal basis $\{\psi_k\}$ in $H$, for example $(\psi_k)_n = \delta_n^k$.

Let $W(t)$ be a Wiener process defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$, taking values in $H$ and with covariance operator $Q$. This means that $W$ is Gaussian, has independent time increments and that for $s, t \geq 0$, $f, g \in H$,

$$\mathbb{E}(W(s), f) = 0 \quad \text{and} \quad \mathbb{E}(W(s), f)(W(t), g) = \big(s \wedge t\big)(Qf, g).$$

Let $\beta_j$ be standard (scalar) mutually independent Wiener processes, $\{e_j\}$ be an orthonormal basis in $H$ consisting of eigen-elements of $Q$, with $Qe_j = q_j e_j$. Then $W$ has the following representation

$$W(t) = \lim_{n\to\infty} W_n(t) \;\text{ in }\; L^2(\Omega; H) \;\text{ with }\; W_n(t) = \sum_{1\leq j\leq n} q_j^{1/2} \beta_j(t) e_j, \tag{2.15}$$

and $Trace(Q) = \sum_{j\geq 1} q_j$. For details concerning this Wiener process see e.g. [13].

Given a viscosity coefficient $\nu > 0$, consider the following stochastic shell model

$$d_t u(t) + \big[\nu A u(t) + B(u(t))\big]\, dt = \sqrt{\nu}\, \sigma_\nu(t, u(t))\, dW(t), \tag{2.16}$$

where the noise intensity $\sigma_\nu : [0,T] \times V \to L_Q(H_0, H)$ of the stochastic perturbation is properly normalized by the square root of the viscosity coefficient $\nu$. We assume that $\sigma_\nu$ satisfies the following growth and Lipschitz conditions:

**Condition (C1):** *$\sigma_\nu \in \mathscr{C}\big([0,T] \times V; L_Q(H_0, H)\big)$, and there exist non negative constants $K_i$ and $L_i$ such that for every $t \in [0,T]$ and $u, v \in V$:*
**(i)** $|\sigma_\nu(t,u)|^2_{L_Q} \leq K_0 + K_1 |u|^2 + K_2 \|u\|^2$,
**(ii)** $|\sigma_\nu(t,u) - \sigma_\nu(t,v)|^2_{L_Q} \leq L_1 |u-v|^2 + L_2 \|u-v\|^2$.

For technical reasons, in order to prove a large deviation principle for the distribution of the solution to (2.16) as the viscosity coefficient $\nu$ converges to 0, we will need some precise estimates on the solution of the equation deduced from (2.16) by shifting the Brownian $W$ by some random element of its RKHS. This cannot be deduced from similar ones on $u$ by means of a Girsanov transformation since the Girsanov density is not uniformly bounded when the intensity of the noise tends to zero (see e.g. [16] or [10]).

To describe a set of admissible random shifts, we introduce the class $\mathscr{A}$ as the set of $H_0-$valued $(\mathscr{F}_t)-$predictable stochastic processes $h$ such that $\int_0^T |h(s)|_0^2 ds < \infty$, a.s. For fixed $M > 0$, let

$$S_M = \Big\{ h \in L^2(0,T; H_0) : \int_0^T |h(s)|_0^2 ds \leq M \Big\}.$$

The set $S_M$, endowed with the following weak topology, is a Polish (complete separable metric) space (see e.g. [5]): $d_1(h,k) = \sum_{k=1}^{\infty} \frac{1}{2^k} \Big| \int_0^T \big(h(s) - k(s), \tilde{e}_k(s)\big)_0 ds \Big|$, where $\{\tilde{e}_k(s)\}_{k=1}^{\infty}$ is an orthonormal basis for $L^2([0,T],H_0)$. For $M>0$ set

$$\mathscr{A}_M = \{h \in \mathscr{A} : h(\omega) \in S_M, \ a.s.\}. \tag{2.17}$$

In order to define the stochastic control equation, we introduce for $\nu \geq 0$ a family of intensity coefficients $\tilde{\sigma}_\nu$ of a random element $h \in \mathscr{A}_M$ for some $M > 0$. The case $\nu = 0$ will be that of an inviscid limit "deterministic" equation with no stochastic integral and which can be dealt with for fixed $\omega$. We assume that for any $\nu \geq 0$ the coefficient $\tilde{\sigma}_\nu$ satisfies the following condition:

**Condition (C2):** $\tilde{\sigma}_\nu \in \mathscr{C}\big([0,T] \times V; L(H_0,H)\big)$ *and there exist constants* $\tilde{K}_{\mathscr{H}}$, $\tilde{K}_i$, *and* $\tilde{L}_j$, *for* $i = 0,1$ *and* $j = 1,2$ *such that:*

$$|\tilde{\sigma}_\nu(t,u)|^2_{L(H_0,H)} \leq \tilde{K}_0 + \tilde{K}_1 |u|^2 + \nu \tilde{K}_{\mathscr{H}} \|u\|^2_{\mathscr{H}}, \quad \forall t \in [0,T], \ \forall u \in V, \tag{2.18}$$

$$|\tilde{\sigma}_\nu(t,u) - \tilde{\sigma}_\nu(t,v)|^2_{L(H_0,H)} \leq \tilde{L}_1 |u-v|^2 + \nu \tilde{L}_2 \|u-v\|^2, \quad \forall t \in [0,T], \ \forall u,v \in V, \tag{2.19}$$

*where* $\mathscr{H} = \mathscr{H}_{\frac{1}{4}}$ *is defined by* (2.2) *and* $|\cdot|_{L(H_0,H)}$ *denotes the (operator) norm in the space* $L(H_0,H)$ *of all bounded linear operators from* $H_0$ *into* $H$. *Note that if* $\nu = 0$ *the previous growth and Lipschitz on* $\tilde{\sigma}_0(t,.)$ *can be stated for* $u,v \in H$.

**Remark 2.1.** Unlike (**C1**) the hypotheses concerning the control intensity coefficient $\tilde{\sigma}_\nu$ involve a weaker topology (we deal with the operator norm $|\cdot|_{L(H_0,H)}$ instead of the trace class norm $|\cdot|_{L_Q}$). However we require in (2.18) a stronger bound (in the interpolation space $\mathscr{H}$). One can see that the noise intensity $\sqrt{\nu}\,\sigma_\nu$ satisfies Condition (**C2**) provided that in Condition (**C1**), we replace point (i) by $|\sigma_\nu(t,u)|^2_{L_Q} \leq K_0 + K_1|u|^2 + K_{\mathscr{H}} \|u\|^2_{\mathscr{H}}$. Thus the class of intensities satisfying both Conditions (**C1**) and (**C2**) when multiplied by $\sqrt{\nu}$ is wider than that those coefficients which satisfy condition **(C1)** with $K_2 = 0$.

Let $M>0$, $h \in \mathscr{A}_M$, $\xi$ an $H$-valued random variable independent of $W$ and $\nu > 0$. Under Conditions (**C1**) and (**C2**) we consider the nonlinear SPDE

$$du_h^\nu(t) + \big[\nu A u_h^\nu(t) + B\big(u_h^\nu(t)\big)\big]\, dt = \sqrt{\nu}\,\sigma_\nu(t, u_h^\nu(t))\, dW(t) + \tilde{\sigma}_\nu(t, u_h^\nu(t)) h(t)\, dt, \tag{2.20}$$

with initial condition $u_h^\nu(0) = \xi$. Using [10], Theorem 3.1, we know that for every $T > 0$ and $\nu > 0$ there exists $\bar{K}_2^\nu := \bar{K}_2(\nu, T, M) > 0$ such that if $h_\nu \in \mathscr{A}_M$, $\mathbb{E}|\xi|^4 < +\infty$ and $0 \leq K_2 < \bar{K}_2^\nu$, equation (2.20) has a unique solution $u_h^\nu \in \mathscr{C}([0,T],H) \cap L^2([0,T],V)$ which satisfies:

$$\begin{aligned}(u_h^\nu, v) - (\xi, v) &+ \int_0^t \big[\nu \langle u_h^\nu(s), Av\rangle + \langle B(u_h^\nu(s)), v\rangle\big]\, ds \\ &= \int_0^t \big(\sqrt{\nu}\,\sigma_\nu(s, u_h^\nu(s))\, dW(s)\,,\, v\big) + \int_0^t \big(\tilde{\sigma}_\nu(s, u_h^\nu(s)) h(s)\,,\, v\big)\, ds\end{aligned}$$

a.s. for all $v \in Dom(A)$ and $t \in [0,T]$. Note that $u_h^\nu$ is a weak solution from the analytical point of view, but a strong one from the probabilistic point of view, that is written in terms of the given

Brownian motion $W$. Furthermore, if $K_2 \in [0, \bar{K}_2^\nu[$ and $L_2 \in [0,2[$, there exists a constant $C_\nu := C(K_i, L_j, \tilde{K}_i, \tilde{K}_{\mathscr{H}}, T, M, \nu)$ such that

$$\mathbb{E}\Big( \sup_{0\le t\le T} |u_h^\nu(t)|^4 + \int_0^T \|u_h^\nu(t)\|^2\, dt + \int_0^T \|u_h^\nu(t)\|_{\mathscr{H}}^4\, dt \Big) \le C_\nu\, (1 + \mathbb{E}|\xi|^4). \tag{2.21}$$

The following proposition proves that $\bar{K}_2^\nu$ can be chosen independent of $\nu$ and that a proper formulation of upper estimates of the $H$, $\mathscr{H}$ and $V$ norms of the solution $u_h^\nu$ to (2.20) can be proved uniformly in $h \in \mathscr{A}_M$ and in $\nu \in (0, \nu_0]$ for some constant $\nu_0 > 0$.

**Proposition 2.2.** *Fix $M > 0$, $T > 0$, $\sigma_\nu$ and $\tilde{\sigma}_\nu$ satisfy Conditions (**C1**)–(**C2**) and let the initial condition $\xi$ be such that $\mathbb{E}|\xi|^4 < +\infty$. Then in any shell model where $B$ is defined by (2.5) or (2.6), there exist constants $\nu_0 > 0$, $\bar{K}_2$ and $\bar{C}(M)$ such that if $0 < \nu \le \nu_0$, $0 \le K_2 < \bar{K}_2$, $L_2 < 2$ and $h \in \mathscr{A}_M$, the solution $u_h^\nu$ to (2.20) satisfies:*

$$\mathbb{E}\Big( \sup_{0\le t\le T} |u_h^\nu(t)|^4 + \nu \int_0^T \|u_h^\nu(s)\|^2\, ds + \nu \int_0^T \|u_h^\nu(s)\|_{\mathscr{H}}^4\, ds \Big) \le \bar{C}(M)\, \big(\mathbb{E}|\xi|^4 + 1\big). \tag{2.22}$$

*Proof.* For every $N > 0$, set $\tau_N = \inf\{t : |u_h^\nu(t)| \ge N\} \wedge T$. Itô's formula and the antisymmetry relation in (2.8) yield that for $t \in [0, T]$,

$$\begin{aligned} |u_h^\nu(t\wedge\tau_N)|^2 = |\xi|^2 + 2\sqrt{\nu} \int_0^{t\wedge\tau_N} \big(\sigma_\nu(s, u_h^\nu(s)) dW(s), u_h^\nu(s)\big) - 2\nu \int_0^{t\wedge\tau_N} \|u_h^\nu(s)\|^2 ds \\ + 2\int_0^{t\wedge\tau_N} \big(\tilde{\sigma}_\nu(s, u_h^\nu(s)) h(s), u_h^\nu(s)\big)\, ds + \nu \int_0^{t\wedge\tau_N} |\sigma_\nu(s, u_h^\nu(s))|_{L_Q}^2\, ds, \end{aligned} \tag{2.23}$$

and using again Itô's formula we have

$$|u_h^\nu(t\wedge\tau_N)|^4 + 4\nu \int_0^{t\wedge\tau_N} |u_h^\nu(r)|^2\, \|u_h^\nu(r)\|^2\, dr \le |\xi|^4 + I(t) + \sum_{1\le j\le 3} T_j(t), \tag{2.24}$$

where

$$\begin{aligned} I(t) &= 4\sqrt{\nu}\, \Big| \int_0^{t\wedge\tau_N} \big(\sigma_\nu(r, u_h^\nu(r))\, dW(r), u_h^\nu(r)\, |u_h^\nu(r)|^2\big) \Big|, \\ T_1(t) &= 4 \int_0^{t\wedge\tau_N} |(\tilde{\sigma}_\nu(r, u_h^\nu(r))\, h(r)\,,\, u_h^\nu(r))|\, |u_h^\nu(r)|^2 dr, \\ T_2(t) &= 2\nu \int_0^{t\wedge\tau_N} |\sigma_\nu(r, u_h^\nu(r))|_{L_Q}^2\, |u_h^\nu(r)|^2 dr, \\ T_3(t) &= 4\nu \int_0^{t\wedge\tau_N} |\sigma_\nu^*(s, u_h^\nu(r))\, u_h^\nu(r)|_0^2\, dr. \end{aligned}$$

Since $h \in \mathscr{A}_M$, the Cauchy-Schwarz and Young inequalities and condition **(C2)** imply that for any $\epsilon > 0$,

$$T_1(t) \le 4 \int_0^{t\wedge\tau_N} \Big( \sqrt{\tilde{K}_0} + \sqrt{\tilde{K}_1}\, |u_h^\nu(r)| + \sqrt{\nu\, \tilde{K}_{\mathscr{H}}}\, k_0^{-\frac{1}{2}} \|u_h^\nu(r)\| \Big)\, |h(r)|_0\, |u_h^\nu(r)|^3 dr$$

$$\leq 4\sqrt{\tilde{K}_0 M T} + 4\Big(\sqrt{\tilde{K}_0} + \sqrt{\tilde{K}_1}\Big) \int_0^{t\wedge\tau_N} |h(r)|_0\, |u_h^\nu(r)|^4\, ds$$

$$+ \epsilon\, \nu \int_0^t \|u_h^\nu(r)\|^2\, |u_h^\nu(r)|^2\, dr + \frac{4\tilde{K}_{\mathcal{H}}}{\epsilon\, k_0} \int_0^{t\wedge\tau_N} |h(r)|_0^2\, |u_h^\nu(r)|^4\, dr. \qquad (2.25)$$

Using condition **(C1)** we deduce

$$T_2(t) + T_3(t) \leq 6\nu \int_0^{t\wedge\tau_N} \big[K_0 + K_1\, |u_h^\nu(r)|^2 + K_2 \|u_h^\nu(r)\|^2\big]\, |u_h^\nu(r)|^2\, dr$$

$$\leq 6\nu\, K_0\, T + 6\nu\, (K_0 + K_1) \int_0^{t\wedge\tau_N} |u_h^\nu(r)|^4\, dr + 6\nu\, K_2 \int_0^t \|u_h^\nu(r)\|^2\, |u_h^\nu(r)|^2 dr. \qquad (2.26)$$

Let $K_2 \leq \frac{1}{2}$ and $0 < \epsilon \leq 2 - 3K_2$; set

$$\varphi(r) = 4\Big(\sqrt{\tilde{K}_0} + \sqrt{\tilde{K}_1}\Big)\, |h(r)|_0 + \frac{4\tilde{K}_{\mathcal{H}}}{\epsilon k_0}\, |h(r)|_0^2 + 6\nu(K_0 + K_1).$$

Then a.s.

$$\int_0^T \varphi(r)\, dr \leq 4\Big(\sqrt{\tilde{K}_0} + \sqrt{\tilde{K}_1}\Big)\sqrt{M\, T} + \frac{4\tilde{K}_{\mathcal{H}}}{\epsilon k_0} M + 6\nu(K_0 + K_1)\, T := \Phi \qquad (2.27)$$

and the inequalities (2.24)-(2.26) yield that for

$$X(t) = \sup_{r\leq t} |u_h^\nu(r\wedge\tau_N)|^4\,,\ Y(t) = \nu \int_0^t \|u_h^\nu(r\wedge\tau_N)\|^2\, |u_h^\nu(r\wedge\tau_N)|^2\, ds,$$

$$X(t) + (4 - 6K_2 - \epsilon) Y(t) \leq |\xi|^4 + \Big(4\sqrt{\tilde{K}_0 M T} + 6\nu K_0 T\Big) + I(t) + \int_0^t \varphi(s) X(s)\, ds. \qquad (2.28)$$

The Burkholder-Davis-Gundy inequality, (**C1**), Cauchy-Schwarz and Young's inequalities yield that for $t \in [0,T]$ and $\delta, \kappa > 0$,

$$\mathbb{E} I(t) \leq 12\sqrt{\nu}\, \mathbb{E}\Big(\Big\{\int_0^{t\wedge\tau_N} \big[K_0 + K_1\, |u_h^\nu(s)|^2 + K_2\, \|u_h^\nu(s)\|^2\big]\, |u_h^\nu(r)|^6 ds\Big\}^{\frac{1}{2}}\Big)$$

$$\leq 12\sqrt{\nu}\, \mathbb{E}\Big(\sup_{0\leq s\leq t} |u_h^\nu(s\wedge\tau_N)|^2 \Big\{\int_0^{t\wedge\tau_N} \big[K_0 + K_1\, |u_h^\nu(s)|^2 + K_2\, \|u_h^\nu(s)\|^2\big]\, |u_h^\nu(s)|^2 ds\Big\}^{\frac{1}{2}}\Big)$$

$$\leq \delta\, \mathbb{E}(Y(t)) + \Big(\frac{36K_2}{\delta} + \kappa\, \nu\Big)\, \mathbb{E}(X(t)) + \frac{36}{\kappa}\Big[K_0\, T + (K_0 + K_1)\int_0^t \mathbb{E}(X(s))\, ds\Big]. \qquad (2.29)$$

Thus we can apply Lemma 3.2 in [10] (see also Lemma 3.2 in [16]), and we deduce that for $0 < \nu \leq \nu_0$, $K_2 \leq \frac{1}{2}$, $\epsilon = \alpha = \frac{1}{2}$, $\beta = \frac{36K_2}{\delta} + \kappa\, \nu_0 \leq 2^{-1} e^{-\Phi}$, $\delta \leq \alpha 2^{-1} e^{-\Phi}$ and $\gamma = \frac{36}{\kappa}(K_0 + K_1)$,

$$\mathbb{E}\Big(X(T) + \alpha Y(T)\Big) \leq 2\exp\big(\Phi + 2T\gamma e^{\Phi}\big)\Big[4\sqrt{\tilde{K}_0\, M\, T} + 6\nu_0 K_0 T + \frac{36}{\kappa} K_0 T + \mathbb{E}(|\xi|^4)\Big]. \qquad (2.30)$$

Using the last inequality from (2.4), we deduce that for $K_2$ small enough, $\bar{C}(M)$ independent of $N$ and $\nu \in ]0, \nu_0]$,

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |u_h^\nu(t\wedge\tau_N)|^4 + \nu \int_0^{\tau_N} \|u_h^\nu(t)\|_{\mathcal{H}}^4\, dt\Big) \leq \bar{C}(M)(1 + \mathbb{E}(|\xi|^4)).$$

As $N \to +\infty$, the monotone convergence theorem yields that for $\bar{K}_2$ small enough and $\nu \in ]0,\nu_0]$

$$\mathbb{E}\Big(\sup_{0\le t\le T} |u_h^\nu(t)|^4 + \nu \int_0^T \|u_h^\nu(t)\|_{\mathcal{H}}^4\, dt\Big) \le \bar{C}(M)(1+\mathbb{E}(|\xi|^4)).$$

This inequality and (2.30) with $t$ instead of $t\wedge\tau_N$ conclude the proof of (2.22) by a similar simpler computation based on conditions **(C1)** and **(C2)**. □

# 3 Well posedeness, more a priori bounds and inviscid equation

The aim of this section is twofold. On one hand, we deal with the inviscid case $\nu = 0$ for which the PDE

$$du_h^0(t) + B(u_h^0(t))\, dt = \tilde{\sigma}_0(t, u_h^0(t))h(t)\, dt\,, \quad u_h^0(0) = \xi \tag{3.1}$$

can be solved for every $\omega$. In order to prove that (3.1) has a unique solution in $\mathscr{C}([0,T],V)$ a.s., we will need stronger assumptions on the constants $\mu, a, b$ defining $B$, the initial condition $\xi$ and $\tilde{\sigma}_0$. The initial condition $\xi$ has to belong to $V$ and the coefficients $a, b, \mu$ have to be chosen such that $(B(u,u),Au) = 0$ for $u \in V$ (see (2.13)). On the other hand, under these assumptions and under stronger assumptions on $\sigma_\nu$ and $\tilde{\sigma}_\nu$, similar to that imposed on $\tilde{\sigma}_0$, we will prove further properties of $u_h^\nu$ for a strictly positive viscosity coefficient $\nu$.

Thus, suppose furthermore that for $\nu > 0$ (resp. $\nu = 0$), the map

$$\tilde{\sigma}_\nu : [0,T]\times Dom(A) \to L(H_0, V)\, (\text{resp. } \tilde{\sigma}_0 : [0,T]\times V \to L(H_0,V))$$

satisfies the following:

**Condition (C3):** *There exist non negative constants $\tilde{K}_i$ and $\tilde{L}_j$, $i = 0,1,2$, $j = 1,2$ such that for $s \in [0,T]$ and for any $u, v \in Dom(A)$ if $\nu > 0$ (resp. for any $u, v \in V$ if $\nu = 0$),*

$$|A^{\frac12}\tilde{\sigma}_\nu(s,u)|^2_{L(H_0,H)} \le \tilde{K}_0 + \tilde{K}_1\, \|u\|^2 + \nu\, \tilde{K}_2\, |Au|^2, \tag{3.2}$$

*and*

$$|A^{\frac12}\tilde{\sigma}_\nu(s,u) - A^{\frac12}\tilde{\sigma}_\nu(s,v)|^2_{L(H_0,H)} \le \tilde{L}_1\, \|u-v\|^2 + \nu\, \tilde{L}_2\, |Au - Av|^2. \tag{3.3}$$

**Theorem 3.1.** *Suppose that $\tilde{\sigma}_0$ satisfies the conditions* **(C2)** *and* **(C3)** *and that the coefficients $a, b, \mu$ defining $B$ satisfy $a(1+\mu^2) + b\mu^2 = 0$. Let $\xi \in V$ be deterministic. For any $M > 0$ there exists $C(M)$ such that equation* (3.1) *has a unique solution in $\mathscr{C}([0,T],V)$ for any $h \in \mathscr{A}_M$, and a.s. one has:*

$$\sup_{h\in\mathscr{A}_M}\ \sup_{0\le t\le T} \|u_h^0(t)\| \le C(M)(1+\|\xi\|). \tag{3.4}$$

Since equation (3.1) can be considered for any fixed $\omega$, it suffices to check that the deterministic equation (3.1) has a unique solution in $\mathscr{C}([0,T],V)$ for any $h \in S_M$ and that (3.4) holds. For any $m \ge 1$, let $H_m = span(\varphi_1, \cdots, \varphi_m) \subset Dom(A)$,

$$P_m : H \to H_m \quad \text{denote the orthogonal projection from } H \text{ onto } H_m, \tag{3.5}$$

and finally let $\tilde{\sigma}_{0,m} = P_m \tilde{\sigma}_0$. Clearly $P_m$ is a contraction of $H$ and $|\tilde{\sigma}_{0,m}(t,u)|^2_{L(H_0,H)} \le |\tilde{\sigma}_0(t,u)|^2_{L(H_0,H)}$. Set $u^0_{m,h}(0) = P_m\,\xi$ and consider the ODE on the $m$-dimensional space $H_m$ defined by

$$d\big(u^0_{m,h}(t)\,,\,v\big) = \big[ -\big(B(u^0_{m,h}(t))\,,\,v\big) + \big(\tilde{\sigma}_0(t,u^0_{m,h}(t))h(t)\,,v\big)\big]\,dt \tag{3.6}$$

for every $v \in H_m$.

Note that using (2.9) we deduce that the map $u \in H_m \mapsto \langle B(u)\,,\,v\rangle$ is locally Lipschitz. Furthermore, since there exists some constant $C(m)$ such that $\|u\| \vee \|u\|_{\mathcal{H}} \le C(m)|u|$ for $u \in H_m$, Condition **(C2)** implies that the map $u \in H_m \mapsto \big((\tilde{\sigma}_{0,m}(t,u)h(t)\,,\,\varphi_k) : 1 \le k \le m\big)$, is globally Lipschitz from $H_m$ to $\mathbb{R}^m$ uniformly in $t$. Hence by a well-known result about existence and uniqueness of solutions to ODEs, there exists a maximal solution $u^0_{m,h} = \sum_{k=1}^m (u^0_{m,h}\,,\,\varphi_k)\,\varphi_k$ to (3.6), i.e., a (random) time $\tau^0_{m,h} \le T$ such that (3.6) holds for $t < \tau^0_{m,h}$ and as $t \uparrow \tau^0_{m,h} < T$, $|u^0_{m,h}(t)| \to \infty$. The following lemma provides the (global) existence and uniqueness of approximate solutions as well as their uniform a priori estimates. This is the main preliminary step in the proof of Theorem 3.1.

**Lemma 3.2.** *Suppose that the assumptions of Theorem 3.1 are satisfied and fix $M > 0$. Then for every $h \in \mathcal{A}_M$ equation (3.6) has a unique solution in $\mathcal{C}([0,T],H_m)$. There exists some constant $C(M)$ such that for every $h \in \mathcal{A}_M$,*

$$\sup_m \sup_{0\le t\le T} \|u^0_{m,h}(t)\|^2 \le C(M)(1 + \|\xi\|^2)\ a.s. \tag{3.7}$$

*Proof.* The proof is included for the sake of completeness; the arguments are similar to that in the classical viscous framework. Let $h \in \mathcal{A}_M$ and let $u^0_{m,h}(t)$ be the approximate maximal solution to (3.6) described above. For every $N > 0$, set $\tau_N = \inf\{t :\ \|u^0_{m,h}(t)\| \ge N\} \wedge T$. Let $\Pi_m : H_0 \to H_0$ denote the projection operator defined by $\Pi_m u = \sum_{k=1}^m (u\,,\,e_k)\,e_k$, where $\{e_k, k \ge 1\}$ is the orthonormal basis of $H$ made by eigen-elements of the covariance operator $Q$ and used in (2.15).

Since $\varphi_k \in Dom(A)$ and $V$ is a Hilbert space, $P_m$ contracts the $V$ norm and commutes with $A$. Thus, using **(C3)** and (2.13), we deduce

$$\begin{aligned}
\|u^0_{m,h}(t\wedge\tau_N)\|^2 &\le \|\xi\|^2 - 2\int_0^{t\wedge\tau_N} \big(B(u^0_{m,h}(s))\,,\,Au^0_{m,h}(s)\big)\,ds \\
&\quad + 2\int_0^{t\wedge\tau_N} \Big|A^{\frac12} P_m \tilde{\sigma}_{0,m}(s,u^0_{m,h}(s))h(s)\Big|\,\|u^0_{m,h}(s)\|\,ds \\
&\le |\xi\|^2 + 2\sqrt{\tilde{K}_0 MT} + 2\Big(\sqrt{\tilde{K}_0} + \sqrt{\tilde{K}_1}\Big)\int_0^{t\wedge\tau_N} |h(s)|_0\,\|u^0_{m,h}(s)\|^2\,ds.
\end{aligned}$$

Since the map $\|u^0_{m,h}(.)\|$ is bounded on $[0,\tau_N]$, Gronwall's lemma implies that for every $N > 0$,

$$\sup_m \sup_{t\le\tau_N} \|u^0_{m,h}(t)\|^2 \le \Big(\|\xi\|^2 + 2\sqrt{\tilde{K}_0 MT}\Big)\,\exp\Big(2\sqrt{MT}\,\Big[\sqrt{\tilde{K}_0} + \sqrt{\tilde{K}_1}\Big]\Big). \tag{3.8}$$

Let $\tau := \lim_N \tau_N$ ; as $N \to \infty$ in (3.8) we deduce

$$\sup_m \sup_{t\le\tau} \|u^0_{m,h}(t)\|^2 \le \Big(\|\xi\|^2 + 2\sqrt{\tilde{K}_0 MT}\Big)\,\exp\Big(2\sqrt{MT}\,\Big[\sqrt{\tilde{K}_0} + \sqrt{\tilde{K}_1}\Big]\Big). \tag{3.9}$$

On the other hand, $\sup_{t\le\tau} \|u^0_{m,h}(t)\|^2 = +\infty$ if $\tau < T$, which contradicts the estimate (3.9) . Hence $\tau = T$ a.s. and we get (3.7) which completes the proof of the Lemma. □

We now prove the main result of this section.

**Proof of Theorem 3.1:**
**Step 1:** Using the estimate (3.7) and the growth condition (2.18) we conclude that each component of the sequence $\big((u^0_{m,h})_n, n\geq 1\big)$ satisfies the following estimate

$$\sup_m \sup_{0\leq t\leq T} |(u^0_{m,h})_n(t)|^2 + \Big|\big(\tilde\sigma_0(t,u^0_{m,h}(t))h(t)\big)_n\Big| \leq C \; a.s., \forall n=1,2,\cdots$$

for some constant $C>0$ depending only on $M, \|\xi\|, T$. Moreover, writing the equation (3.1) for the GOY shell model in the componentwise form using (2.5) (the proof for the Sabra shell model using (2.6), which is similar, is omitted), we obtain for $n=1,2,\cdots$

$$\begin{aligned}(u^0_{m,h})_n(t) =& (P_m\xi)_n + i\int_0^t \big(ak_{n+1}(u^0_{m,h})^*_{n+1}(s)(u^0_{m,h})^*_{n+2}(s) + bk_n(u^0_{m,h})^*_{n-1}(s)(u^0_{m,h})^*_{n+1}(s)\\ &- ak_{n-1}(u^0_{m,h})^*_{n-1}(s)(u^0_{m,h})^*_{n-2}(s) - bk_{n-1}(u^0_{m,h})^*_{n-2}(s)(u^0_{m,h})^*_{n-1}(s)\big)ds\\ &+\int_0^t \big(\tilde\sigma_0(s,u^0_{m,h}(s))\,h(s)\big)_n ds\,. \end{aligned}\tag{3.10}$$

Hence, we deduce that for every $n\geq 1$ there exists a constant $C_n$, independent of $m$, such that

$$\|(u^0_{m,h})_n\|_{C^1([0,T];\mathbb{C})} \leq C_n.$$

Applying the Ascoli-Arzelà theorem, we conclude that for every $n$ there exists a subsequence $(m^n_k)_{k\geq 1}$ such that $(u^0_{m^n_k,h})_n$ converges uniformly to some $(u^0_h)_n$ as $k\longrightarrow\infty$. By a diagonal procedure, we may choose a sequence $(m^n_k)_{k\geq 1}$ independent of $n$ such that $(u^0_{m,h})_n$ converges uniformly to some $(u^0_h)_n\in\mathscr{C}([0,T];\mathbb{C})$ for every $n\geq 1$; set

$$u^0_h(t) = ((u^0_h)_1, (u^0_h)_2, \ldots).$$

From the estimate (3.7), we have the weak star convergence in $L^\infty(0,T;V)$ of some further subsequence of $\big(u^0_{m^n_k,h} : k\geq 1\big)$. The weak limit belongs to $L^\infty(0,T;V)$ and has clearly $(u^0_h)_n$ as components that belong to $\mathscr{C}([0,T];\mathbb{C})$ for every integer $n\geq 1$. Using the uniform convergence of each component, it is easy to show, passing to the limit in the expression (3.10), that $u^0_h(t)$ satisfies the weak form of the GOY shell model equation (3.1). Finally, since

$$u^0_h(t) = \xi + \int_0^t \big[-B(u^0_h(s)) + \tilde\sigma_0(s,u^0_h(s))h(s)\big]\,ds,$$

is such that $\sup_{0\leq s\leq T}\|u^0_h(s)\|<\infty$ a.s. and since for every $s\in[0,T]$, by (2.9) and (3.2) we have a.s.

$$\big[\|B(u^0_h(s))\| + \|\tilde\sigma_0(s,u^0_h(s))h(s)\|\big] \leq C\Big(1+\sup_{0\leq s\leq T}\|u^0_h(s)\|^2\big)\big(1+|h(s)|_0\Big) \in L^2([0,T]),$$

we deduce that $u^0_h\in\mathscr{C}([0,T],V)$ a.s.

**Step 2:** To complete the proof of Theorem 3.1, we show that the solution $u^0_h$ to (3.1) is unique in $\mathscr{C}([0,T],V)$. Let $v\in\mathscr{C}([0,T],V)$ be another solution to (3.1) and set

$$\tau_N = \inf\{t\geq 0 : \|u^0_h(t)\|\geq N\}\wedge\inf\{t\geq 0 : \|v(t)\|\geq N\}\wedge T.$$

Since $\|u_h^0(.)\|$ and $\|v(.)\|$ are bounded on $[0,T]$, we have $\tau_N \to T$ as $N \to \infty$.
Set $U = u_h^0 - v$; equation (2.10) implies

$$\begin{aligned}\Big|\big(A^{\frac12}B(u_h^0(s)) - A^{\frac12}B(v(s)), A^{\frac12}U(s)\big)\Big| &= \Big|\big(B(u_h^0(s)) - B(v(s)), AU(s)\big)\Big| \\ &\le \bar{C}_1 \|U(s)\|^2\, \|v(s)\|.\end{aligned}$$

On the other hand, the Lipschitz property (3.3) from condition **(C3)** for $\nu = 0$ implies

$$\Big|\big[A^{\frac12}\tilde{\sigma}_0(s,u_h^0(s)) - A^{\frac12}\tilde{\sigma}_0(s,v(s))\big]h(s)\Big| \le \sqrt{\tilde{L}_1}\|u_h^0(s) - v(s)\|\, |h(s)|_0.$$

Therefore,

$$\begin{aligned}\|U(t\wedge\tau_N)\|^2 \;&= \int_0^{t\wedge\tau_N} \Big\{ -2\Big(A^{\frac12}B(u_h^0(s)) - A^{\frac12}B(v(s)), A^{\frac12}U(s)\Big) \\ &\qquad + 2\Big([A^{\frac12}\tilde{\sigma}_0(s,u_h^0(s)) - A^{\frac12}\tilde{\sigma}_0(s,v(s))]h(s), A^{\frac12}U(s)\Big)\Big\}\, ds \\ &\le 2\int_0^t \big(\bar{C}_1 N + \sqrt{L}_1|h(s)|_0\big)\, \|U(s\wedge\tau_N)\|^2\, ds,\end{aligned}$$

and Gronwall's lemma implies that (for almost every $\omega$) $\sup_{0\le t\le T} \|U(t\wedge\tau_N)\|^2 = 0$ for every $N$. As $N\to\infty$, we deduce that a.s. $U(t) = 0$ for every $t$, which concludes the proof. □

We now suppose that the diffusion coefficient $\sigma_\nu$ satisfies the following condition **(C4)** which strengthens **(C1)** in the way **(C3)** strengthens **(C2)**, i.e.,

**Condition (C4):** *There exist constants $K_i$ and $L_i$, $i = 0,1,2$, $j = 1,2$, such that for any $\nu > 0$ and $u \in Dom(A)$,*

$$|A^{\frac12}\sigma_\nu(s,u)|^2_{L_Q} \;\le\; K_0 + K_1\|u\|^2 + K_2|Au|^2, \tag{3.11}$$

$$|A^{\frac12}\sigma_\nu(s,u) - A^{\frac12}\sigma_\nu(s,v)|^2_{L_Q} \;\le\; L_1\|u-v\|^2 + L_2|Au - Av|^2. \tag{3.12}$$

Then for $\nu > 0$, the existence result and apriori bounds of the solution to (2.20) proved in Proposition 2.2 can be improved as follows.

**Proposition 3.3.** *Let $\xi \in V$, let the coefficients $a, b, \mu$ defining $B$ be such that $a(1+\mu^2) + b\mu^2 = 0$, $\sigma_\nu$ and $\tilde{\sigma}_\nu$ satisfy the conditions* **(C1)***,* **(C2)***,* **(C3)** *and* **(C4)***. Then there exist positive constants $\bar{K}_2$ and $\nu_0$ such that for $0 < K_2 < \bar{K}_2$ and $0 < \nu \le \nu_0$, for every $M > 0$ there exists a constant $C(M)$ such that for any $h \in \mathscr{A}_M$, the solution $u_h^\nu$ to (2.20) belongs to $\mathscr{C}([0,T],V)$ almost surely and*

$$\sup_{h\in\mathscr{A}_M}\ \sup_{0<\nu\le\nu_0} \mathbb{E}\Big(\sup_{t\in[0,T]} \|u_h^\nu(t)\|^2 + \nu\int_0^T |Au_h^\nu(t)|^2\, dt\Big) \le C(M). \tag{3.13}$$

*Proof.* Fix $m \ge 1$, let $P_m$ be defined by (3.5) and let $u^\nu_{m,h}(t)$ be the approximate maximal solution to the (finite dimensional) evolution equation: $u^\nu_{m,h}(0) = P_m\xi$ and

$$du^\nu_{m,h}(t) \;=\; \big[-\nu P_m A u^\nu_{m,h}(t) - P_m B(u^\nu_{m,h}(t)) + P_m\tilde{\sigma}_\nu(t, u^\nu_{m,h}(t))h(t)\big]\, dt$$

$$+P_m\sqrt{\nu}\,\sigma_\nu(t,u^\nu_{m,h})(t)dW_m(t), \tag{3.14}$$

where $W_m$ is defined by (2.15). Proposition 3.3 in [10] proves that (3.14) has a unique solution $u^\nu_{m,h}\in\mathscr{C}([0,T],P_m(H))$. For every $N>0$, set

$$\tau_N=\inf\{t:\ \|u^\nu_{m,h}(t)\|\geq N\}\wedge T.$$

Since $P_m(H)\subset Dom(A)$, we may apply Itô's formula to $\|u^\nu_{m,h}(t)\|^2$. Let $\Pi_m:H_0\to H_0$ be defined by $\Pi_m u=\sum_{k=1}^m\big(u,e_k\big)\,e_k$ for some orthonormal basis $\{e_k,k\geq 1\}$ of $H$ made by eigen-vectors of the covariance operator $Q$; then we have:

$$\begin{aligned}
&\|u^\nu_{m,h}(t\wedge\tau_N)\|^2=\|P_m\xi\|^2+2\sqrt{\nu}\int_0^{t\wedge\tau_N}\big(A^{\frac12}P_m\sigma_\nu(s,u^\nu_{m,h}(s))dW_m(s),A^{\frac12}u^\nu_{m,h}(s)\big)\\
&+\nu\int_0^{t\wedge\tau_N}|P_m\sigma_\nu(s,u^\nu_{m,h}(s))\,\Pi_m|^2_{L_Q}\,ds-2\int_0^{t\wedge\tau_N}\big(A^{\frac12}B(u^\nu_{m,h}(s))\,,\,A^{\frac12}u^\nu_{m,h}(s)\big)\,ds\\
&-2\nu\int_0^{t\wedge\tau_N}\big(A^{\frac12}P_mAu^\nu_{m,h}(s),A^{\frac12}u^\nu_{m,h}(s)\big)\,ds+2\int_0^{t\wedge\tau_N}\big(A^{\frac12}P_m\tilde\sigma_\nu(s,u^\nu_{m,h}(s))h(s),A^{\frac12}u^\nu_{m,h}(s)\big)\,ds.
\end{aligned}$$

Since the functions $\varphi_k$ are eigen-functions of $A$, we have $A^{\frac12}P_m=P_mA^{\frac12}$ and hence $\big(A^{\frac12}P_mAu^\nu_{m,h}(s),A^{\frac12}u^\nu_{m,h}(s)\big)=|Au^\nu_{m,h}(s)|^2$. Furthermore, $P_m$ contracts the $H$ and the $V$ norms, and for $u\in Dom(A)$, $\big(B(u),Au\big)=0$ by (2.13). Hence for $0<\epsilon=\frac12(2-K_2)<1$, using Cauchy-Schwarz's inequality and the conditions **(C3)** and **(C4)** on the coefficients $\sigma_\nu$ and $\tilde\sigma_\nu$, we deduce

$$\begin{aligned}
&\|u^\nu_{m,h}(t\wedge\tau_N)\|^2+\epsilon\nu\int_0^{t\wedge\tau_N}\big|Au^\nu_{m,h}(s)|^2\,ds\leq\|\xi\|^2+\nu\int_0^{t\wedge\tau_N}\big[K_0+K_1\|u^\nu_{m,h}(s)\|^2\big]\,ds\\
&\quad+2\sqrt{\nu}\int_0^{t\wedge\tau_N}\big(A^{\frac12}P_m\sigma_\nu(s,u^\nu_{m,h}(s))dW_m(s),A^{\frac12}u^\nu_{m,h}(s)\big)\\
&\quad+2\int_0^{t\wedge\tau_N}\Big\{\Big[\sqrt{\tilde K_0}+\Big(\sqrt{\tilde K_0}+\sqrt{\tilde K_1}\Big)\|u^\nu_{m,h}(s)\|^2\Big]|h(s)|_0+\frac{\tilde K_2}{\epsilon}|h(s)|_0^2\|u^\nu_{m,h}(s)\|^2\Big\}\,ds.
\end{aligned}$$

For any $t\in[0,T]$ set

$$\begin{aligned}
I(t)&=\sup_{0\leq s\leq t}\Big|2\sqrt{\nu}\int_0^{s\wedge\tau_N}\big(A^{\frac12}P_m\sigma_\nu(r,u^\nu_{m,h}(r))dW_m(r)\,,\,A^{\frac12}u^\nu_{m,h}(r)\big)\Big|,\\
X(t)&=\sup_{0\leq s\leq t}\|u^\nu_{m,h}(s\wedge\tau_N)\|^2,\quad Y(t)=\int_0^{t\wedge\tau_N}|Au^\nu_{m,h}(r)|^2\,dr,\\
\varphi(t)&=2\Big(\sqrt{\tilde K_0}+\sqrt{\tilde K_1}\Big)\,|h(t)|_0+\nu K_1+\frac{\tilde K_2}{\epsilon}|h(t)|_0^2.
\end{aligned}$$

Then almost surely, $\int_0^T\varphi(t)\,dt\leq\nu K_1T+2\big(\sqrt{\tilde K_0}+\sqrt{\tilde K_1}\big)\sqrt{MT}+\frac{\tilde K_2}{\epsilon}M:=C$. The Burkholder-Davis-Gundy inequality, conditions (**C1**) – (**C4**), Cauchy-Schwarz and Young's inequalities yield that for $t\in[0,T]$ and $\beta>0$,

$$\mathbb{E}I(t)\;\leq\;6\sqrt{\nu}\,\mathbb{E}\Big\{\int_0^{t\wedge\tau_N}\big|A^{\frac12}\sigma_\nu(s,u^\nu_{m,h}(r))\,\Pi_m|^2_{L_Q}\,\|u^\nu_{m,h}(s)\|^2ds\Big\}^{\frac12}$$

$$\leq \beta\, \mathbb{E}\Big( \sup_{0\leq s\leq t\wedge\tau_N} \|u_{m,h}(s)\|^2\Big) + \frac{9\nu K_1}{\beta}\, \mathbb{E}\int_0^{t\wedge\tau_N} \|u_{m,h}(s)\|^2\, ds$$
$$+ \frac{9\nu K_0}{\beta}\, T + \frac{9\nu K_2}{\beta}\mathbb{E}\int_0^{t\wedge\tau_N} |Au^\nu_{m,h}(s)|^2 ds.$$

Set $Z = \|\xi\|^2 + \nu_0 K_0 T + 2\sqrt{\tilde{K}_0 TM}$, $\alpha = \epsilon\nu$, $\beta = 2^{-1}e^{-C}$, $K_2 < 2^{-2}e^{-2C}(9+2^{-3}e^{-2C})^{-1}$; the previous inequality implies that the bounded function $X$ satisfies a.s. the inequality

$$X(t) + \alpha Y(t) \leq Z + I(t) + \int_0^t \varphi(s) X(s)\, ds.$$

Furthermore, $I(t)$ is non decreasing, such that for $0 < \nu \leq \nu_0$, $\delta = \frac{9\nu K_2}{\beta} \leq \alpha 2^{-1} e^{-C}$ and $\gamma = \frac{9\nu_0}{a} K_1$, one has

$$\mathbb{E}I(t) \leq \beta\mathbb{E}X(t) + \gamma\mathbb{E}\int_0^t X(s)\, ds + \delta Y(t) + \frac{9\nu_0}{\beta} K_0 T.$$

Lemma 3.2 from [10] implies that for $K_2$ and $\nu_0$ small enough, there exists a constant $C(M,T)$ which does not depend on $m$ and $N$, and such that for $0 < \nu \leq \nu_0$, $m \geq 1$ and $h \in \mathscr{A}_M$:

$$\sup_{N>0}\sup_{m\geq 1} \mathbb{E}\Big[ \sup_{0\leq t\leq\tau_N} \|u^\nu_{m,h}(t)\|^2 + \nu\int_0^{\tau_N} |Au^\nu_{m,h}(t)|^2\, dt\Big] < \infty.$$

Then, letting $N \to \infty$ and using the monotone convergence theorem, we deduce that

$$\sup_{m\geq 1}\sup_{h\in\mathscr{A}_M} \mathbb{E}\Big[ \sup_{0\leq t\leq T} \|u^\nu_{m,h}(t)\|^2 + \nu\int_0^{T} |Au^\nu_{m,h}(t)|^2\, dt\Big] < \infty. \tag{3.15}$$

Then using classical arguments we prove the existence of a subsequence of $(u^\nu_{m,h}, m \geq 1)$ which converges weakly in $L^2([0,T]\times\Omega, V) \cap L^4([0,T]\times\Omega, \mathscr{H})$ and converges weak-star in $L^4(\Omega, L^\infty([0,T], H))$ to the solution $u^\nu_h$ to equation (2.20) (see e.g. [10], proof of Theorem 3.1). In order to complete the proof, it suffices to extract a further subsequence of $(u^\nu_{m,h}, m \geq 1)$ which is weak-star convergent to the same limit $u^\nu_h$ in $L^2(\Omega, L^\infty([0,T], V))$ and converges weakly in $L^2(\Omega\times[0,T], Dom(A))$; this is a straightforward consequence of (3.15). Then as $m \to \infty$ in (3.15), we conclude the proof of (3.13). □

# 4 Large deviations

We will prove a large deviation principle using a weak convergence approach [4; 5], based on variational representations of infinite dimensional Wiener processes. Let $\sigma : [0,T]\times V \to L_Q$ and for every $\nu > 0$ let $\bar{\sigma}_\nu : [0,T]\times Dom(A) \to L_Q$ satisfy the following condition:

**Condition (C5):**
*(i) There exist a positive constant $\gamma$ and non negative constants $\bar{C}$, $\bar{K}_0$, $\bar{K}_1$ and $\bar{L}_1$ such that for all $u, v \in V$ and $s, t \in [0,T]$:*

$$|\sigma(t,u)|^2_{L_Q} \leq \bar{K}_0 + \bar{K}_1\, |u|^2, \quad \Big|A^{\frac{1}{2}}\sigma(t,u)\Big|^2_{L_Q} \leq \bar{K}_0 + \bar{K}_1\, \|u\|^2,$$

$$|\sigma(t,u)-\sigma(t,v)|^2_{L_Q} \le \bar{L}_1\, |u-v|^2, \quad \Big|A^{\frac12}\sigma(t,u)-A^{\frac12}\sigma(t,v)\Big|^2_{L_Q} \le \bar{L}_1\, \|u-v\|^2,$$
$$\Big|\sigma(t,u)-\sigma(s,u)\Big|_{L_Q} \le C\,(1+\|u\|)\,|t-s|^{\gamma}.$$

*(ii) There exist a positive constant $\gamma$ and non negative constants $\bar{C}$, $\bar{K}_0$, $\bar{K}_{\mathscr{H}}$, $\bar{K}_2$ and $\bar{L}_2$ such that for $\nu>0$, $s,t\in[0,T]$ and $u,v\in Dom(A)$,*

$$|\bar{\sigma}_\nu(t,u)|^2_{L_Q} \le \big(\bar{K}_0+\bar{K}_{\mathscr{H}}\,\|u\|^2_{\mathscr{H}}\big), \quad \Big|A^{\frac12}\bar{\sigma}_\nu(t,u)\Big|^2_{L_Q} \le \big(\bar{K}_0+\bar{K}_2\,|Au|^2\big),$$
$$|\bar{\sigma}_\nu(t,u)-\bar{\sigma}_\nu(t,v)|^2_{L_Q} \le \bar{L}_2\,\|u-v\|^2, \Big|A^{\frac12}\bar{\sigma}_\nu(t,u)-A^{\frac12}\bar{\sigma}_\nu(t,v)\Big|^2_{L_Q} \le \bar{L}_2\,|Au-Av|^2,$$
$$\Big|\bar{\sigma}_\nu(t,u)-\bar{\sigma}_\nu(t,u)\Big|_{L_Q} \le \bar{C}\,(1+\|u\|)\,|t-s|^{\gamma}.$$

Set

$$\sigma_\nu=\tilde{\sigma}_\nu=\sigma+\sqrt{\nu}\,\bar{\sigma}_\nu \;\text{ for }\quad \nu>0, \quad \text{and} \quad \tilde{\sigma}_0=\sigma. \tag{4.1}$$

Then for $0\le\nu\le\nu_1$, the coefficients $\sigma_\nu$ and $\tilde{\sigma}_\nu$ satisfy the conditions **(C1)-(C4)** with

$$\begin{aligned} & K_0=\tilde{K}_0=4\bar{K}_0,\; K_1=\tilde{K}_1=2\bar{K}_1,\; L_1=\tilde{L}_1=2\bar{L}_1,\; \tilde{K}_2=2\bar{K}_2,\; \tilde{K}_{\mathscr{H}}=2\bar{K}_{\mathscr{H}},\\ & K_2=2\big[\bar{K}_2\vee\big(\bar{K}_{\mathscr{H}}k_0^{4\alpha-2}\big)\big]\nu_1,\; L_2=2\bar{L}_2\nu_1 \quad \text{and } \tilde{L}_2=2\bar{L}_2. \end{aligned} \tag{4.2}$$

Proposition 3.3 and Theorem 3.1 prove that for some $\nu_0\in]0,\nu_1]$, $\bar{K}_2$ and $\bar{L}_2$ small enough, $0<\nu\le\nu_0$ (resp. $\nu=0$), $\xi\in V$ and $h_\nu\in\mathscr{A}_M$, the following equation has a unique solution $u^\nu_{h_\nu}$ (resp. $u^0_h$) in $\mathscr{C}(0,T],V)$: $u^\nu_{h_\nu}(0)=u^0_h(0)=\xi$, and

$$du^\nu_{h_\nu}(t)+\big[\nu Au^\nu_{h_\nu}(t)+B(u^\nu_{h_\nu}(t))\big]dt=\sqrt{\nu}\,\sigma_\nu(t,u^\nu_{h_\nu}(t))\,dW(t)+\tilde{\sigma}_\nu(t,u^\nu_{h_\nu}(t))h_\nu(t)dt, \tag{4.3}$$
$$du^0_h(t)+B(u^0_h(t))\,dt=\sigma(t,u^0_h(t))\,h(t)\,dt. \tag{4.4}$$

Recall that for any $\alpha\ge0$, $\mathscr{H}_\alpha$ has been defined in (2.2) and is endowed with the norm $\|\cdot\|_\alpha$ defined in (2.2). When $0\le\alpha\le\frac14$, as $\nu\to0$ we will establish a Large Deviation Principle (LDP) in the set $\mathscr{C}([0,T],V)$ for the uniform convergence in time when $V$ is endowed with the norm $\|\cdot\|_\alpha$ for the family of distributions of the solutions $u^\nu$ to the evolution equation: $u^\nu(0)=\xi\in V$,

$$du^\nu(t)+\big[\nu Au^\nu(t)+B(u^\nu(t))\big]\,dt=\sqrt{\nu}\sigma_\nu(t,u^\nu(t))\,dW(t), \tag{4.5}$$

whose existence and uniqueness in $\mathscr{C}([0,T],V)$ follows from Propositions 2.2 and 3.3. Unlike in [27], [16], [22] and [10], the large deviations principle is not obtained in the natural space, which is here $\mathscr{C}([0,T],V)$ under the assumptions **(C5)**, because the lack of viscosity does not allow to prove that $u^0_h(t)\in Dom(A)$ for almost every $t$.

To obtain the LDP in the best possible space with the weak convergence approach, we need an extra condition, which is part of condition **(C5)** when $\alpha=0$, that is when $\mathscr{H}_\alpha=H$.

**Condition (C6):** *Let $\alpha\in[0,\frac14]$; there exists a constant $L_3$ such that for $u,v\in\mathscr{H}_\alpha$ and $t\in[0,1]$,*

$$\Big|A^\alpha\sigma(t,u)-A^\alpha\sigma(t,v)|_{L_Q}\le L_3\,\|u-v\|_\alpha. \tag{4.6}$$

Let $\mathscr{B}$ denote the Borel $\sigma-$field of the Polish space

$$\mathscr{X} = \mathscr{C}([0,T],V) \quad \text{endowed with the norm} \quad \|u\|_{\mathscr{X}} =: \sup_{0\le t\le T} \|u(t)\|_{\alpha}, \tag{4.7}$$

where $\|\cdot\|_{\alpha}$ is defined by (2.2). We at first recall some classical definitions; by convention the infimum over an empty set is $+\infty$.

**Definition 4.1.** *The random family $(u^\nu)$ is said to satisfy a large deviation principle on $\mathscr{X}$ with the good rate function $I$ if the following conditions hold:*
*$I$* ***is a good rate function.*** *The function $I : \mathscr{X} \to [0,\infty]$ is such that for each $M \in [0,\infty[$ the level set $\{\phi \in \mathscr{X} : I(\phi) \le M\}$ is a compact subset of $\mathscr{X}$.*
*For $A \in \mathscr{B}$, set $I(A) = \inf_{u\in A} I(u)$.*
***Large deviation upper bound.*** *For each closed subset $F$ of $\mathscr{X}$:*

$$\limsup_{\nu\to 0}\ \nu \log \mathbb{P}(u^\nu \in F) \le -I(F).$$

***Large deviation lower bound.*** *For each open subset $G$ of $\mathscr{X}$:*

$$\liminf_{\nu\to 0}\ \nu \log \mathbb{P}(u^\nu \in G) \ge -I(G).$$

Let $\mathscr{C}_0 = \{\int_0^{\cdot} h(s)ds\ :\ h \in L^2([0,T],H_0)\} \subset \mathscr{C}([0,T],H_0)$. Given $\xi \in V$ define $\mathscr{G}^0_\xi : \mathscr{C}([0,T],H_0) \to X$ by $\mathscr{G}^0_\xi(g) = u^0_h$ for $g = \int_0^{\cdot} h(s)ds \in \mathscr{C}_0$ and $u^0_h$ is the solution to the (inviscid) control equation (4.4) with initial condition $\xi$, and $\mathscr{G}^0_\xi(g) = 0$ otherwise. The following theorem is the main result of this section.

**Theorem 4.2.** *Let $\alpha \in [0,\frac{1}{4}]$, suppose that the constants $a,b,\mu$ defining $B$ are such that $a(1+\mu^2)+b\mu^2 = 0$, let $\xi \in V$, and let $\sigma_\nu$ and $\tilde{\sigma}_\nu$ be defined for $\nu > 0$ by* (4.1) *with coefficients $\sigma$ and $\bar{\sigma}_\nu$ satisfying the conditions* (**C5**) *and* **(C6)** *for this value of $\alpha$. Then the solution $(u^\nu)_{\nu>0}$ to* (4.5) *with initial condition $\xi$ satisfies a large deviation principle in $\mathscr{X} := \mathscr{C}([0,T],V)$ endowed with the norm $\|u\|_{\mathscr{X}} =: \sup_{0\le t\le T} \|u(t)\|_\alpha$, with the good rate function*

$$I(u) = \inf_{\{h\in L^2(0,T;H_0):\ u=\mathscr{G}^0_\xi(\int_0^{\cdot} h(s)ds)\}} \Big\{\frac{1}{2}\int_0^T |h(s)|_0^2\, ds\Big\}. \tag{4.8}$$

We at first prove the following technical lemma, which studies time increments of the solution to the stochastic control problem (4.3) which extends both (4.5) and (4.4).

To state this lemma, we need the following notations. For every integer $n$, let $\psi_n : [0,T] \to [0,T]$ denote a measurable map such that: $s \le \psi_n(s) \le \big(s + c2^{-n}\big) \wedge T$ for some positive constant $c$ and for every $s \in [0,T]$. Given $N > 0$, $h_\nu \in \mathscr{A}_M$, for $t \in [0,T]$ and $\nu \in [0,\nu_0]$, let

$$G^\nu_N(t) = \Big\{\omega\ :\ \Big(\sup_{0\le s\le t} \|u^\nu_h(s)(\omega)\|^2\Big) \vee \Big(\int_0^t |Au^\nu_h(s)(\omega)|^2 ds\Big) \le N\Big\}.$$

**Lemma 4.3.** *Let $a,b,\mu$ satisfy the condition $a(1+\mu^2)+b\mu^2 = 0$. Let $\nu_0, M, N$ be positive constants, $\sigma$ and $\bar{\sigma}_\nu$ satisfy condition* **(C5)***, $\sigma_\nu$ and $\tilde{\sigma}_\nu$ be defined by* (4.1) *for $\nu \in [0,\nu_0]$. For every $\nu \in ]0,\nu_0]$, let $\xi \in L^4(\Omega;H) \cap L^2(\Omega;V)$, $h_\nu \in \mathscr{A}_M$ and let $u^\nu_{h_\nu}(t)$ denote solution to* (4.3)*. For $\nu = 0$, let $\xi \in V$,*

*$h \in \mathscr{A}_M$, let $u_h^0(t)$ denote be solution to (4.4). Then there exists a positive constant $C$ (depending on $K_i, \tilde{K}_i, L_i, \tilde{L}_i, T, M, N, \nu_0$) such that:*

$$I_n(h_\nu, \nu) := \mathbb{E}\Big[ 1_{G_N^\nu(T)} \int_0^T \|u_{h_\nu}^\nu(s) - u_{h_\nu}^\nu(\psi_n(s))\|^2 \, ds \Big] \le C\, 2^{-\frac{n}{2}} \ \text{ for } 0 < \nu \le \nu_0, \tag{4.9}$$

$$I_n(h, 0) := 1_{G_N^0(T)} \int_0^T \|u_h^0(s) - u_h^0(\psi_n(s))\|^2 \, ds \le C\, 2^{-n} \ \text{ a.s. for } \nu = 0. \tag{4.10}$$

*Proof.* For $\nu > 0$, the proof is close to that of Lemma 4.2 in [16]. Let $\nu \in ]0, \nu_0]$, $h \in \mathscr{A}_M$,; for any $s \in [0, T]$, Itô's formula yields

$$\|u_{h_\nu}^\nu(\psi_n(s)) - u_{h_\nu}^\nu(s)\|^2 = 2 \int_s^{\psi_n(s)} \big(A[u_{h_\nu}^\nu(r) - u_{h_\nu}^\nu(s)], du_{h_\nu}^\nu(r)\big) + \nu \int_s^{\psi_n(s)} |A^{\frac{1}{2}} \sigma(r, u_{h_\nu}^\nu(r))|_{L_Q}^2 dr.$$

Therefore $I_n(h_\nu, \nu) = \sum_{1 \le i \le 5} I_{n,i}(h_\nu, \nu)$, where

$$\begin{aligned}
I_{n,1}(h_\nu, \nu) &= 2\sqrt{\nu}\, \mathbb{E}\Big( 1_{G_N^\nu(T)} \int_0^T ds \int_s^{\psi_n(s)} \big(A^{\frac{1}{2}} \sigma_\nu(r, u_{h_\nu}^\nu(r)) dW(r), A^{\frac{1}{2}} [u_{h_\nu}^\nu(r) - u_{h_\nu}^\nu(s)]\big) \Big),\\
I_{n,2}(h_\nu, \nu) &= \nu\, \mathbb{E}\Big( 1_{G_N^\nu(T)} \int_0^T ds \int_s^{\psi_n(s)} |A^{\frac{1}{2}} \sigma_\nu(r, u_{h_\nu}^\nu(r))|_{L_Q}^2 \, dr \Big),\\
I_{n,3}(h_\nu, \nu) &= -2\, \mathbb{E}\Big( 1_{G_N^\nu(T)} \int_0^T ds \int_s^{\psi_n(s)} \big\langle A^{\frac{1}{2}} B(u_{h_\nu}^\nu(r)), A^{\frac{1}{2}} [u_{h_\nu}^\nu(r) - u_{h_\nu}^\nu(s)] \big\rangle \, dr \Big),\\
I_{n,4}(h_\nu, \nu) &= -2\nu\, \mathbb{E}\Big( 1_{G_N^\nu(T)} \int_0^T ds \int_s^{\psi_n(s)} \big\langle A^{\frac{3}{2}} u_{h_\nu}^\nu(r), A^{\frac{1}{2}} [u_{h_\nu}^\nu(r) - u_{h_\nu}^\nu(s)] \big\rangle \, dr \Big),\\
I_{n,5}(h_\nu, \nu) &= 2\, \mathbb{E}\Big( 1_{G_N^\nu(T)} \int_0^T ds \int_s^{\psi_n(s)} \big(A^{\frac{1}{2}} \tilde{\sigma}_\nu(r, u_{h_\nu}^\nu(r)) h_\nu(r), A^{\frac{1}{2}} [u_{h_\nu}^\nu(r) - u_{h_\nu}^\nu(s)]\big) \, dr \Big).
\end{aligned}$$

Clearly $G_N^\nu(T) \subset G_N^\nu(r)$ for $r \in [0, T]$. Furthermore, $\|u_h^\nu(r)\|^2 \vee \|u_h^\nu(s)\|^2 \le N$ on $G_N^\nu(r)$ for $0 \le s \le r \le T$.

The Burkholder-Davis-Gundy inequality and (**C5**) yield for $0 < \nu \le \nu_0$

$$\begin{aligned}
|I_{n,1}(h_\nu, \nu)| &\le 6\sqrt{\nu} \int_0^T ds\, \mathbb{E}\Big( \int_s^{\psi_n(s)} \big|A^{\frac{1}{2}} \sigma_\nu(r, u_{h_\nu}^\nu(r))\big|_{L_Q}^2 1_{G_N^\nu(r)} \|u_{h_\nu}^\nu(r) - u_{h_\nu}^\nu(s)\|^2 \, dr \Big)^{\frac{1}{2}}\\
&\le 6\sqrt{2\nu_0 N} \int_0^T ds\, \mathbb{E}\Big( \int_s^{\psi_n(s)} \big[K_0 + K_1 \|u_{h_\nu}^\nu(r)\|^2 + K_2 |Au_{h_\nu}^\nu(r)|^2\big] \, dr \Big)^{\frac{1}{2}}.
\end{aligned}$$

The Cauchy-Schwarz inequality and Fubini theorem as well as (3.13), which holds uniformly in $\nu \in ]0, \nu_0]$ for small enough fixed $\nu_0 > 0$, imply

$$\begin{aligned}
|I_{n,1}(h_\nu, \nu)| &\le 6\sqrt{2\nu_0 N T} \Big[ \mathbb{E} \int_0^T \big[K_0 + K_1 \|u_{h_\nu}^\nu(r)\|^2 + K_2 |Au_{h_\nu}^\nu(r)|^2\big] \Big( \int_{(r - c2^{-n}) \vee 0}^r ds \Big) dr \Big]^{\frac{1}{2}}\\
&\le C_1 \sqrt{N}\, 2^{-\frac{n}{2}}
\end{aligned} \tag{4.11}$$

for some constant $C_1$ depending only on $K_i$, $i=0,1,2$, $L_j$, $j=1,2$, $M$, $\nu_0$ and $T$. The property (**C5**) and Fubini's theorem imply that for $0<\nu\le\nu_0$,

$$\begin{aligned}|I_{n,2}(h_\nu,\nu)| &\le \nu\,\mathbb{E}\Big(1_{G_N^\nu(T)}\int_0^T ds\int_s^{\psi_n(s)}\big[K_0+K_1\|u^\nu_{h_\nu}(r)\|^2+K_2|Au^\nu_{h_\nu}(r)|^2\big]\,dr\Big)\\ &\le \nu_0\mathbb{E}\int_0^T\big[K_0+K_1\,\|u^\nu_{h_\nu}(r)\|^2+K_2|Au^\nu_{h_\nu}(r)|^2\big]\,c2^{-n}\,dr\;\le\;C_1 2^{-n}\end{aligned}\tag{4.12}$$

for some constant $C_1$ as above. Since $\langle B(u),Au\rangle=0$ and $\|B(u)\|\le C\|u\|^2$ for $u\in V$ by (2.9), we deduce that

$$\begin{aligned}|I_{n,3}(h_\nu,\nu)| &\le 2\mathbb{E}\Big(1_{G_N^\nu(T)}\int_0^T ds\int_s^{\psi_n(s)} dr\,\big(A^{\frac12}B(u^\nu_{h_\nu}(r))\,,\,A^{\frac12}u^\nu_{h_\nu}(s)\big)\Big)\\ &\le 2C\mathbb{E}\Big(1_{G_N^\nu(T)}\int_0^T ds\int_s^{\psi_n(s)}\|u^\nu_{h_\nu}(r)\|^2\,\|u^\nu_h(s)\|\,dr\Big)\le 2C\,N^{\frac32}\,T^2 2^{-n}.\end{aligned}\tag{4.13}$$

Using Cauchy-Schwarz's inequality and (3.13) we deduce that

$$\begin{aligned}I_{n,4}(h_\nu,\nu) &\le 2\nu\,\mathbb{E}\Big(1_{G_N^\nu(T)}\int_0^T ds\int_s^{\psi_n(s)}dr\big[-|Au^\nu_{h_\nu}(r)|^2+|Au^\nu_{h_\nu}(r)|\,|Au^\nu_{h_\nu}(s)|\big]\Big)\\ &\le \frac{\nu}{2}\,\mathbb{E}\Big(\int_0^T ds\,|Au^\nu_{h_\nu}(s)|^2\int_s^{\psi_n(s)}dr\Big)\le C_1 2^{-n}\end{aligned}\tag{4.14}$$

for some constant $C_1$ as above. Finally, Cauchy-Schwarz's inequality, Fubini's theorem, (**C5**) and the definition of $\mathscr{A}_M$ yield

$$\begin{aligned}|I_{n,5}(h_\nu,\nu)| &\le 2\,\mathbb{E}\Big(1_{G_N^\nu(T)}\int_0^T ds\int_s^{\psi_n(s)}dr\\ &\qquad\big[\tilde K_0+\tilde K_1\|u^\nu_{h_\nu}(r)\|^2+\nu\tilde K_2|Au^\nu_{h_\nu}(r)|^2\big]^{\frac12}\,|h_\nu(r)|_0\,\|u^\nu_{h_\nu}(r)-u^\nu_{h_\nu}(s)\|\Big)\\ &\le 4\sqrt{N}\,\mathbb{E}\Big(1_{G_N^\nu(T)}\,(\tilde K_0+\tilde K_1N)^{\frac12}\int_0^T|h_\nu(r)|_0\Big(\int_{(r-c2^{-n})\vee 0}^r ds\Big)\,dr\Big)\\ &\quad+4\sqrt{N}\mathbb{E}\Big(1_{G_N^\nu(T)}\sqrt{\nu_0\tilde K_2}\int_0^T|Au^\nu_{h_\nu}(r)|\,|h_\nu(r)|_0\Big(\int_{(r-c2^{-n})\vee 0}^r ds\Big)\,dr\Big)\\ &\le 4\sqrt{N}\Big[\sqrt{MT}\,(\tilde K_0+\tilde K_1N)^{\frac12}+\big(\nu_0\tilde K_2\,NM\big)^{\frac12}\Big]\,c\,T2^{-n}\le C(\nu_0,N,M,T)\,2^{-n}.\end{aligned}\tag{4.15}$$

Collecting the upper estimates from (4.11)-(4.15), we conclude the proof of (4.9) for $0<\nu\le\nu_0$.

Let $h\in\mathscr{A}_M$; a similar argument for $\nu=0$ yields for almost every $\omega$

$$1_{G_N^0(T)}\int_0^T\int_0^T\|u^0_h(\psi_n(s))-u^0_h(s)\|^2\,ds\le\sum_{j=1,2}I_{n,j}(h,0),$$

with

$$I_{n,1}(h,0)\;=\;-2\,1_{G_N^0(T)}\int_0^T ds\int_s^{\psi_n(s)}\big\langle A^{\frac12}B(u^0_h(r))\,,\,A^{\frac12}\big[u^0_h(r)-u^0_h(s)\big]\big\rangle\,dr,$$

$$I_{n,2}(h,0) \;=\; 2\,1_{G_N^0(T)}\int_0^T ds\int_s^{\psi_n(s)}\big(A^{\frac12}\tilde\sigma_0(r,u_h^\nu(r))h(r)\,,A^{\frac12}\big[u_h^\nu(r)-u_h^\nu(s)\big]\big)\,dr.$$

An argument similar to that which gives (4.13) proves

$$|I_{n,1}(h,0)|\le C(T,N)\,2^{-n}. \tag{4.16}$$

Cauchy-Schwarz's inequality and **(C5)** imply

$$\begin{aligned}|I_{n,2}(h,0)| &\le 2\,1_{G_N^0(T)}\int_0^T ds\int_s^{\psi_n(s)} dr\,\big(\tilde K_0+\tilde K_1\|u_h^0(r)\|^2\big)^{\frac12}\,|h(r)|_0\,\|u_h^0(r)-u_h^0(s)\|\\ &\le 4\sqrt{N}\,\big(\tilde K_0+\tilde K_1 N\big)^{\frac12}\int_0^T |h(r)|_0\Big(\int_{(r-c2^{-n})\vee 0}^{r} ds\Big)\,dr\le C(N,M,T)\,2^{-n}.\end{aligned} \tag{4.17}$$

The inequalities (4.16) and (4.17) conclude the proof of (4.10). □

Now we return to the setting of Theorem 4.2. Let $\nu_0\in]0,\nu_1]$ be defined by Theorem 2.2 and Proposition 3.3, $(h_\nu, 0<\nu\le\nu_0)$ be a family of random elements taking values in the set $\mathscr{A}_M$ defined by (2.17). Let $u^\nu_{h_\nu}$ be the solution of the corresponding stochastic control equation (4.3) with initial condition $u^\nu_{h_\nu}(0)=\xi\in V$. Note that $u^\nu_{h_\nu}=\mathscr{G}^\nu_\xi\Big(\sqrt{\nu}\big(W_.+\frac{1}{\sqrt{\nu}}\int_0^{\cdot}h_\nu(s)ds\big)\Big)$ due to the uniqueness of the solution. The following proposition establishes the weak convergence of the family $(u^\nu_{h_\nu})$ as $\nu\to 0$. Its proof is similar to that of Proposition 4.5 in [10]; see also Proposition 3.3 in [16].

**Proposition 4.4.** *Let $a,b,\mu$ be such that $a(1+\mu^2)+b\mu^2=0$. Let $\alpha\in[0,\frac14]$, $\sigma$ and $\bar\sigma_\nu$ satisfy the conditions (**C5**) and **(C6)** for this value of $\alpha$, $\sigma_\nu$ and $\tilde\sigma_\nu$ be defined by* (4.1). *Let $\xi$ be $\mathscr{F}_0$-measurable such that $\mathbb{E}\big(|\xi|_H^4+\|\xi\|^2\big)<\infty$, and let $h_\nu$ converge to $h$ in distribution as random elements taking values in $\mathscr{A}_M$, where this set is defined by* (2.17) *and endowed with the weak topology of the space $L_2(0,T;H_0)$. Then as $\nu\to 0$, the solution $u^\nu_{h_\nu}$ of* (4.3) *converges in distribution in $\mathscr{X}$ (defined by* (4.7)*) to the solution $u_h^0$ of* (4.4)*. That is, as $\nu\to 0$, the process $\mathscr{G}^\nu_\xi\Big(\sqrt{\nu}\big(W_.+\frac{1}{\sqrt{\nu}}\int_0^{\cdot}h_\nu(s)ds\big)\Big)$ converges in distribution to $\mathscr{G}^0_\xi\big(\int_0^{\cdot}h(s)ds\big)$ in $\mathscr{C}([0,T],V)$ for the topology of uniform convergence on $[0,T]$ where $V$ is endowed with the norm $\|\cdot\|_\alpha$.*

*Proof.* Since $\mathscr{A}_M$ is a Polish space (complete separable metric space), by the Skorokhod representation theorem, we can construct processes $(\tilde h_\nu,\tilde h,\tilde W)$ such that the joint distribution of $(\tilde h_\nu,\tilde W)$ is the same as that of $(h_\nu,W)$, the distribution of $\tilde h$ coincides with that of $h$, and $\tilde h_\nu\to\tilde h$, a.s., in the (weak) topology of $S_M$. Hence a.s. for every $t\in[0,T]$, $\int_0^t\tilde h_\nu(s)ds-\int_0^t\tilde h(s)ds\to 0$ weakly in $H_0$. To ease notations, we will write $(\tilde h_\nu,\tilde h,\tilde W)=(h_\nu,h,W)$. Let $U_\nu=u^\nu_{h_\nu}-u^0_h\in\mathscr{C}([0,T],V)$; then $U_\nu(0)=0$ and

$$\begin{aligned}dU_\nu(t)=&-\big[\nu Au^\nu_{h_\nu}(t)+B(u^\nu_{h_\nu}(t))-B(u^0_h(t))\big]\,dt+\big[\sigma(t,u^\nu_{h_\nu}(t))h_\nu(t)-\sigma(t,u^0_h(t))h(t)\big]\,dt\\ &+\sqrt{\nu}\,\sigma_\nu(t,u^\nu_{h_\nu}(t))\,dW(t)+\sqrt{\nu}\,\bar\sigma_\nu(t,u^\nu_{h_\nu}(t))\,h_\nu(t)\,dt.\end{aligned} \tag{4.18}$$

On any finite time interval $[0,t]$ with $t\le T$, Itô's formula, yields for $\nu>0$ and $\alpha\in[0,\frac12]$:

$$\|U_\nu(t)\|_\alpha^2=-2\nu\int_0^t\big(A^{1+\alpha}u^\nu_{h_\nu}(s),A^\alpha U_\nu(s)\big)ds-2\int_0^t\big\langle A^\alpha\big[B(u^\nu_{h_\nu}(s))-B(u^0_h(s))\big],A^\alpha U_\nu(s)\big\rangle ds$$

$$+2\sqrt{\nu}\int_0^t \big(A^\alpha\sigma_\nu(s,u^\nu_{h_\nu}(s))dW(s)\,,A^\alpha U_\nu(s)\big)+\nu\int_0^t |A^\alpha\sigma_\nu(s,u^\nu_{h_\nu}(s))|^2_{L_Q}\,ds$$
$$+2\sqrt{\nu}\int_0^t \big(A^\alpha\tilde\sigma_\nu(s,u^\nu_{h_\nu}(s))\,h_\nu(s)\,,A^\alpha U_\nu(s)\big)\,ds$$
$$+2\int_0^t \big(A^\alpha\big[\sigma(s,u^\nu_{h_\nu}(s))h_\nu(s)-\sigma(s,u^0_h(s))\,h(s)\big]\,,A^\alpha U_\nu(s)\big)\,ds.$$

Furthermore, $\big(A^\alpha\tilde\sigma_\nu(s,u^\nu_{h_\nu}(s))h_\nu(s)\,,A^\alpha U_\nu(s)\big)=\big(\tilde\sigma_\nu(s,u^\nu_{h_\nu}(s))h_\nu(s)\,,A^{2\alpha}U_\nu(s)\big)$. Cauchy-Schwarz's inequality, conditions **(C5)** and **(C6)**, (2.12) and (2.4) yield since $\alpha\in[0,\frac14]$

$$\begin{aligned}
\|U_\nu(t)\|^2_\alpha \le{}& 2\nu\int_0^t \Big|A^{\frac12+2\alpha}u^\nu_{h_\nu}(s)\Big|\,\big(\|u^\nu_{h_\nu}(s)\|+\|u^0_h(s)\|\big)\,ds\\
&+2C\int_0^t \|U_\nu(s)\|^2_\alpha\,\big(\|u^\nu_{h_\nu}(s)\|+\|u^0_h(s)\|\big)\,ds+2\sqrt{\nu}\int_0^t\big(\sigma_\nu(s,u^\nu_{h_\nu}(s))dW(s)\,,A^{2\alpha}U_\nu(s)\big)\\
&+\nu\int_0^t \big[K_0+K_1\|u^\nu_{h_\nu}(s)\|^2+K_2|Au^\nu_{h_\nu}(s)|^2\big]\,ds\\
&+2\sqrt{\nu}\int_0^t \Big[\sqrt{\tilde K_0}+k_0^{-\frac12}\sqrt{\tilde K_{\mathscr H}}\|u^\nu_{h_\nu}(s)\|\Big]\,|h_\nu(s)|_0\,k_0^{4\alpha-1}\big(\|u^\nu_{h_\nu}(s)\|+\|u^0_h(s)\|\big)\,ds\\
&+2\int_0^t \big(A^\alpha\big[\sigma(s,u^\nu_{h_\nu}(s))-\sigma(s,u^0_h(s))\big]h_\nu(s)\,,A^\alpha U_\nu(s)\big)\,ds\\
&+2\int_0^t \big(A^\alpha\sigma(s,u^0_h(s))\,\big[h_\nu(s)-h_0(s)\big]\,,A^\alpha U_\nu(s)\big)\,ds\\
\le{}& 2\int_0^t \|U_\nu(s)\|^2_\alpha\,\big[C\|u^\nu_{h_\nu}(s)\|^2+C\|u^0_h(s)\|^2+L_3|h_\nu(s)|_0\big]\,ds+\sum_{1\le j\le 5}T_j(t,\nu),
\end{aligned}\tag{4.19}$$

where using again the fact that $\alpha\le\frac14$, we have

$$\begin{aligned}
T_1(t,\nu)&=2\nu\sup_{s\le t}\big[\|u^\nu_{h_\nu}(s)\|+\|u^0_h(s)\|\big]\int_0^t |Au^\nu_{h_\nu}(s)|\,ds,\\
T_2(t,\nu)&=2\sqrt{\nu}\int_0^t\big(\sigma_\nu(s,u^\nu_{h_\nu}(s))dW(s)\,,A^{2\alpha}U_\nu(s)\big),\\
T_3(t,\nu)&=\nu\int_0^t\big[K_0+K_1\|u^\nu_{h_\nu}(s)\|^2+K_2|Au^\nu_{h_\nu}(s)|^2\big]\,ds,\\
T_4(t,\nu)&=2\sqrt{\nu}\,k_0^{2\alpha-1}\int_0^t\Big[\sqrt{\tilde K_0}+k_0^{-\frac12}\sqrt{\tilde K_{\mathscr H}}\,\|u^\nu_{h_\nu}(s)\|\Big]\,|h_\nu(s)|_0\big(\|u^\nu_{h_\nu}(s)\|+\|u^0_h(s)\|\big)ds,\\
T_5(t,\nu)&=2\int_0^t\Big(\sigma(s,u^0_h(s))\,\big(h_\nu(s)-h(s)\big),A^{2\alpha}U_\nu(s)\Big)\,ds.
\end{aligned}$$

We want to show that as $\nu\to0$, $\sup_{t\in[0,T]}\|U_\nu(s)\|_\alpha\to0$ in probability, which implies that $u^\nu_{h_\nu}\to u^0_h$

in distribution in $X$. Fix $N>0$ and for $t\in[0,T]$ let

$$G_N(t) = \Big\{ \sup_{0\le s\le t} \|u_h^0(s)\|^2 \le N\Big\},$$

$$G_{N,\nu}(t) = G_N(t)\cap \Big\{ \sup_{0\le s\le t} \|u_{h_\nu}^\nu(s)\|^2 \le N\Big\} \cap \Big\{ \nu\int_0^t |Au_{h_\nu}(s)|^2 ds \le N\Big\}.$$

The proof consists in two steps.

**Step 1:** For $\nu_0>0$ given by Proposition 3.3 and Theorem 3.1, we have

$$\sup_{0<\nu\le\nu_0}\ \sup_{h,h_\nu\in\mathscr{A}_M} \mathbb{P}(G_{N,\nu}(T)^c)\to 0 \quad \text{as } N\to+\infty.$$

Indeed, for $\nu\in]0,\nu_0]$, $h,h_\nu\in\mathscr{A}_M$, the Markov inequality and the a priori estimates (3.4) and (3.13), which holds uniformly in $\nu\in]0,\nu_0]$, imply that for $0<\nu\le\nu_0$,

$$\begin{aligned}\mathbb{P}(G_{N,\nu}(T)^c) &\le \frac{1}{N}\sup_{h,h_\nu\in\mathscr{A}_M}\mathbb{E}\Big(\sup_{0\le s\le T}\|u_h^0(s)\|^2+\sup_{0\le s\le T}\|u_{h_\nu}^\nu(s)|^2+\nu\int_0^T |Au_{h_\nu}^\nu(s)|^2\,ds\Big)\\ &\le C\,\big(1+\mathbb{E}|\xi|^4+\mathbb{E}\|\xi\|^2\big)N^{-1},\end{aligned}\tag{4.20}$$

for some constant $C$ depending on $T$ and $M$, but independent of $N$ and $\nu$.

**Step 2:** Fix $N>0$, let $h,h_\nu\in\mathscr{A}_M$ be such that $h_\nu\to h$ a.s. in the weak topology of $L^2(0,T;H_0)$ as $\nu\to 0$. Then one has:

$$\lim_{\nu\to 0}\mathbb{E}\Big[1_{G_{N,\nu}(T)}\sup_{0\le t\le T}\|U_\nu(t)|_\alpha^2\Big]=0.\tag{4.21}$$

Indeed, (4.19) and Gronwall's lemma imply that on $G_{N,\nu}(T)$, one has for $0<\nu\le\nu_0$:

$$\sup_{0\le t\le T}\|U_\nu(t)\|_\alpha^2\le \exp\Big(4NC+2L_3\sqrt{MT}\Big)\sum_{1\le j\le 5}\ \sup_{0\le t\le T}T_j(t,\nu)\,.\tag{4.22}$$

Cauchy-Schwarz's inequality implies that for some constant $C(N,T)$ independent on $\nu$:

$$\begin{aligned}\mathbb{E}\Big(1_{G_{N,\nu}(T)}\sup_{0\le t\le T}|T_1(t,\nu)|\Big) &\le 4\sqrt{TN}\,\sqrt{\nu}\,\mathbb{E}\Big(1_{G_{N,\nu}(T)}\Big\{\int_0^T |Au_{h_\nu}^\nu(s)|^2\,ds\Big\}^{\frac12}\Big)\\ &\le C(N,T)\,\sqrt{\nu}.\end{aligned}\tag{4.23}$$

Since the sets $G_{N,\nu}(.)$ decrease, the Burkholder-Davis-Gundy inequality, $\alpha\le\frac14$, the inequality (2.4) and (**C5**) imply that for some constant $C(N,T)$ independent of $\nu$:

$$\begin{aligned}\mathbb{E}\Big(1_{G_{N,\nu}(T)}\sup_{0\le t\le T}|T_2(t,\nu)|\Big) &\le 6\sqrt{\nu}\,\mathbb{E}\Big\{\int_0^T 1_{G_{N,\nu}(s)}\,k_0^{4(2\alpha-\frac12)}\,\|U_\nu(s)\|^2\,|\sigma_\nu(s,u_{h_\nu}^\nu(s))|_{L_Q}^2 ds\Big\}^{\frac12}\\ \le 6\sqrt{\nu}k_0^{2(2\alpha-\frac12)}\mathbb{E}\Big\{\int_0^T 1_{G_{N,\nu}(s)}4N\,(K_0+K_1\|u_{h_\nu}^\nu(s)\|^2+K_2|Au_{h_\nu}^\nu(s)|^2)ds\Big\}^{\frac12} &\le C(T,N)\sqrt{\nu}.\end{aligned}\tag{4.24}$$

The Cauchy-Schwarz inequality implies

$$\mathbb{E}\Big(1_{G_{N,\nu}(T)}\sup_{0\le t\le T}|T_4(t,\nu)|\Big)\le\sqrt{\nu}\,C(N,M,T).\tag{4.25}$$

The definition of $G_{N,\nu}(T)$ implies that

$$\mathbb{E}\Big(1_{G_{N,\nu}(T)}\sup_{0\le t\le T}|T_3(t,\nu)|\Big)\le C\,T\,N\,\nu. \tag{4.26}$$

The inequalities (4.22) - (4.26) show that the proof of (4.21) reduces to check that

$$\lim_{\nu\to 0}\mathbb{E}\Big(1_{G_{N,\nu}(T)}\sup_{0\le t\le T}|T_5(t,\nu)|\Big)=0. \tag{4.27}$$

In further estimates we use Lemma 4.3 with $\psi_n=\bar{s}_n$, where $\bar{s}_n$ is the step function defined by $\bar{s}_n=kT2^{-n}$ for $(k-1)T2^{-n}\le s<kT2^{-n}$. For any $n,N\ge 1$, if we set $t_k=kT2^{-n}$ for $0\le k\le 2^n$, we obviously have

$$\mathbb{E}\Big(1_{G_{N,\nu}(T)}\sup_{0\le t\le T}|T_5(t,\nu)|\Big)\le 2\sum_{1\le i\le 4}\tilde{T}_i(N,n,\nu)+2\,\mathbb{E}\big(\bar{T}_5(N,n,\nu)\big), \tag{4.28}$$

where

$$\begin{aligned}
\tilde{T}_1(N,n,\nu)=&\mathbb{E}\Big[1_{G_{N,\nu}(T)}\sup_{0\le t\le T}\Big|\int_0^t\Big(\sigma(s,u_h^0(s))\big(h_\nu(s)-h(s)\big)\,,A^{2\alpha}\big[U_\nu(s)-U_\nu(\bar{s}_n)\big]\Big)ds\Big|\Big],\\
\tilde{T}_2(N,n,\nu)=&\mathbb{E}\Big[1_{G_{N,\nu}(T)}\\
&\times\sup_{0\le t\le T}\Big|\int_0^t\Big([\sigma(s,u_h^0(s))-\sigma(\bar{s}_n,u_h^0(s))](h_\nu(s)-h(s))\,,A^{2\alpha}U_\nu(\bar{s}_n)\Big)ds\Big|\Big],\\
\tilde{T}_3(N,n,\nu)=&\mathbb{E}\Big[1_{G_{N,\nu}(T)}\\
&\times\sup_{0\le t\le T}\Big|\int_0^t\Big(\big[\sigma(\bar{s}_n,u_h^0(s))-\sigma(\bar{s}_n,u_h^0(\bar{s}_n))\big]\big(h_\nu(s)-h(s)\big)\,,A^{2\alpha}U_\nu(\bar{s}_n)\Big)ds\Big|\Big],\\
\tilde{T}_4(N,n,\nu)=&\mathbb{E}\Big[1_{G_{N,\nu}(T)}\sup_{1\le k\le 2^n}\sup_{t_{k-1}\le t\le t_k}\Big|\Big(\sigma(t_k,u_h^0(t_k))\int_{t_{k-1}}^t(h_\nu(s)-h(s))ds\,,A^{2\alpha}U_\nu(t_k)\Big)\Big|\Big],\\
\bar{T}_5(N,n,\nu)=&1_{G_{N,\nu}(T)}\sum_{1\le k\le 2^n}\Big|\Big(\sigma(t_k,u_h^0(t_k))\int_{t_{k-1}}^{t_k}\big(h_\nu(s)-h(s)\big)\,ds\,,A^{2\alpha}U_\nu(t_k)\Big)\Big|.
\end{aligned}$$

Using the Cauchy-Schwarz and Young inequalities, (**C5**), (2.4), (4.9) and (4.10) in Lemma 4.3 with $\psi_n(s)=\bar{s}_n$, we deduce that for some constant $\bar{C}_1:=C(T,M,N)$ independent of $\nu\in]0,\nu_0]$,

$$\begin{aligned}
\tilde{T}_1(N,n,\nu)\le{}& k_0^{4\alpha-1}\mathbb{E}\Big[1_{G_{N,\nu}(T)}\int_0^T(\bar{K}_0+\bar{K}_1|u_h^0(s)|^2)^{\frac12}|h_\nu(s)-h(s)|_0\,\big\|U_\nu(s)-U_\nu(\bar{s}_n)\big\|\,ds\Big]\\
\le{}& k_0^{4\alpha-1}\Big(\mathbb{E}\Big[1_{G_{N,\nu}(T)}\int_0^T 2\{\|u_{h_\nu}^\nu(s)-u_{h_\nu}^\nu(\bar{s}_n)\|^2+\|u_h^0(s)-u_h^0(\bar{s}_n)\|^2\}\,ds\Big]\Big)^{\frac12}\\
&\times\sqrt{\bar{K}_0+k_0^{-2}\bar{K}_1N}\,\Big(\mathbb{E}\int_0^T 2\big[|h_\nu(s)|_0^2+|h(s)|_0^2\big]\,ds\Big)^{\frac12}\le\bar{C}_1\,2^{-\frac{n}{4}}.
\end{aligned}\tag{4.29}$$

A similar computation based on (**C5**) and (4.10) from Lemma 4.3 yields for some constant $\bar{C}_3:=C(T,M,N)$ and any $\nu\in]0,\nu_0]$

$$\tilde{T}_3(N,n,\nu)\le\sqrt{2Nk_0^{-2}L_1}\Big(\mathbb{E}\Big[1_{G_{N,\nu}(T)}\int_0^T\|u_h^0(s)-u_h^0(\bar{s}_n)\|^2\,ds\Big]\Big)^{\frac12}\Big(\mathbb{E}\int_0^T|h_\nu(s)-h(s)|_0^2\,ds\Big)^{\frac12}$$

$$\leq \bar{C}_3\, 2^{-\frac{n}{4}}. \tag{4.30}$$

The Hölder regularity **(C5)** imposed on $\sigma(.,u)$ and the Cauchy-Schwarz inequality imply that

$$\tilde{T}_2(N,n,\nu) \leq C\sqrt{N}2^{-n\gamma}\,\mathbb{E}\Big(1_{G_{N,\nu}(T)}\int_0^T \Big(1+\|u_h^0(s)\|\Big)\,|h_\nu(s)-h(s)|_0 ds\Big) \leq \bar{C}_2 2^{-n\gamma} \tag{4.31}$$

for some constant $\bar{C}_2 = C(T,M,N)$. Using Cauchy-Schwarz's inequality and (**C5**) we deduce for $\bar{C}_4 = C(T,N,M)$ and any $\nu \in ]0,\nu_0]$

$$\begin{aligned}\tilde{T}_4(N,n,\nu) &\leq \mathbb{E}\Big[1_{G_{N,\nu}(T)}\sup_{1\leq k\leq 2^n}\big(\bar{K}_0+\bar{K}_1|u_h^0(t_k)|^2\big)^{\frac{1}{2}}\int_{t_{k-1}}^{t_k}|h_\nu(s)-h(s)|_0\,ds\,\|U_\nu(t_k)\|\,k_0^{4\alpha-1}\Big]\\ &\leq C(N)\,\mathbb{E}\Big(\sup_{1\leq k\leq 2^n}\int_{t_{k-1}}^{t_k}\big(|h_\nu(s)|_0+|h(s)|_0\big)\,ds\Big) \leq \bar{C}_4\,2^{-\frac{n}{2}}.\end{aligned} \tag{4.32}$$

Finally, note that the weak convergence of $h_\nu$ to $h$ implies that as $\nu \to 0$, for any $a,b \in [0,T]$, $a<b$, the integral $\int_a^b h_\nu(s)ds \to \int_a^b h(s)ds$ in the weak topology of $H_0$. Therefore, since the operator $\sigma(t_k,u_h^0(t_k))$ is compact from $H_0$ to $H$, we deduce that for every $k$,

$$\Big|\sigma(t_k,u_h^0(t_k))\Big(\int_{t_{k-1}}^{t_k}h_\nu(s)ds - \int_{t_{k-1}}^{t_k}h(s)ds\Big)\Big|_H \to 0 \quad\text{as}\quad \nu\to 0.$$

Hence a.s. for fixed $n$ as $\nu \to 0$, $\bar{T}_5(N,n,\nu) \to 0$ while $\bar{T}_5(N,n,\nu) \leq C(\bar{K}_0,\bar{K}_1,N,n,M)$. The dominated convergence theorem proves that $\mathbb{E}(\bar{T}_5(N,n,\nu)) \to 0$ as $\nu \to 0$ for any fixed $n,N$.

This convergence and (4.28)–(4.32) complete the proof of (4.27). Indeed, they imply that for any fixed $N \geq 1$ and any integer $n \geq 1$

$$\limsup_{\nu\to 0}\mathbb{E}\Big[1_{G_{N,\nu}(T)}\sup_{0\leq t\leq T}|T_5(t,\nu)|\Big] \leq C_{N,T,M}\;2^{-n(\frac{1}{4}\wedge\gamma)}.$$

for some constant $C(N,T,M)$ independent of $n$. Since $n$ is arbitrary, this yields for any integer $N \geq 1$ the convergence property (4.27) holds. By the Markov inequality, we have for any $\delta > 0$

$$\mathbb{P}\Big(\sup_{0\leq t\leq T}\|U_\nu(t)\|_\alpha > \delta\Big) \leq \mathbb{P}(G_{N,\nu}(T)^c) + \frac{1}{\delta^2}\mathbb{E}\Big(1_{G_{N,\nu}(T)}\sup_{0\leq t\leq T}\|U_\nu(t)\|_\alpha^2\Big).$$

Finally, (4.20) and (4.21) yield that for any integer $N \geq 1$,

$$\limsup_{\nu\to 0}\mathbb{P}\Big(\sup_{0\leq t\leq T}\|U_\nu(t)\|_\alpha > \delta) \leq C(T,M,\delta)N^{-1},$$

for some constant $C(T,M,\delta)$ which does not depend on $N$. Letting $N \to +\infty$ concludes the proof of the proposition. □

The following compactness result is the second ingredient which allows to transfer the LDP from $\sqrt{\nu}W$ to $u^\nu$. Its proof is similar to that of Proposition 4.4 and easier; it will be sketched (see also [16], Proposition 4.4).

**Proposition 4.5.** *Suppose that the constants $a, b, \mu$ defining $B$ satisfy the condition $a(1+\mu^2)+b\mu^2 = 0$, $\sigma$ satisfies the conditions (**C5**) and (**C6**) and let $\alpha \in [0, \frac{1}{4}]$. Fix $M > 0$, $\xi \in V$ and let $K_M = \{u_h^0 : h \in S_M\}$, where $u_h^0$ is the unique solution in $\mathscr{C}([0,T],V)$ of the deterministic control equation (4.4). Then $K_M$ is a compact subset of $\mathscr{X} = \mathscr{C}([0,T],V)$ endowed with the norm $\|u\|_{\mathscr{X}} = \sup_{0\le t\le T} \|u(t)\|_\alpha$.*

*Proof.* To ease notation, we skip the superscript 0 which refers to the inviscid case. By Theorem 3.1, $K_M \subset \mathscr{C}([0,T],V)$. Let $\{u_n\}$ be a sequence in $K_M$, corresponding to solutions of (4.4) with controls $\{h_n\}$ in $S_M$:

$$du_n(t) + B(u_n(t))dt = \sigma(t,u_n(t))h_n(t)dt,\;\; u_n(0) = \xi.$$

Since $S_M$ is a bounded closed subset in the Hilbert space $L^2(0,T;H_0)$, it is weakly compact. So there exists a subsequence of $\{h_n\}$, still denoted as $\{h_n\}$, which converges weakly to a limit $h \in L^2(0,T;H_0)$. Note that in fact $h \in S_M$ as $S_M$ is closed. We now show that the corresponding subsequence of solutions, still denoted as $\{u_n\}$, converges in $X$ to $u$ which is the solution of the following "limit" equation

$$du(t) + B(u(t))dt = \sigma(t,u(t))h(t)dt,\;\; u(0) = \xi.$$

Note that we know from Theorem 3.1 that $u \in \mathscr{C}([0,T],V)$, and that one only needs to check that the convergence of $u_n$ to $u$ holds uniformly in time for the weaker $\|\cdot\|_\alpha$ norm on $V$. To ease notation we will often drop the time parameters $s$, $t$, ... in the equations and integrals. Let $U_n = u_n - u$; using (2.12) and **(C6)**, we deduce that for $t \in [0,T]$,

$$\begin{aligned}
\|U_n(t)\|_\alpha^2 &= -2\int_0^t \big(A^\alpha B(u_n(s)) - A^\alpha B(u(s))\,,\, A^\alpha U_n(s)\big)\,ds \\
&\quad + 2\int_0^t \Big\{ \Big( A^\alpha \big[\sigma(s,u_n(s)) - \sigma(s,u(s))\big] h_n(s)\,,\, A^\alpha U_n(s)\Big) \\
&\quad + \big(A^\alpha \sigma(s,u(s))\big(h_n(s)-h(s)\big)\,,\, A^\alpha U_n(s)\big)\Big\} ds \\
&\le 2C\int_0^t \|U_n(s)\|_\alpha^2 \big(\|u_n(s)\| + \|u(s)\|\big)\,ds + 2L_3\int_0^t \|U_n(s)\|_\alpha^2 |h_n(s)|_0\,ds \\
&\quad + 2\int_0^t \Big(\sigma(s,u(s))\,[h_n(s)-h(s)]\;,\; A^{2\alpha} U_n(s)\Big)\,ds.
\end{aligned} \tag{4.33}$$

The inequality (3.4) implies that there exists a finite positive constant $\tilde{C}$ such that

$$\sup_n \sup_{0\le t\le T} \big(\|u(t)\|^2 + \|u_n(t)\|^2\big) = \tilde{C}. \tag{4.34}$$

Thus Gronwall's lemma implies that

$$\sup_{0\le t\le T} \|U_n(t)\|_\alpha^2 \le \exp\Big(2C\tilde{C} + 2L_3\sqrt{MT}\Big) \sum_{1\le i\le 5} I_{n,N}^i, \tag{4.35}$$

where, as in the proof of Proposition 4.4, we have:

$$I_{n,N}^1 \;=\; \int_0^T \Big|\big(\sigma(s,u(s))\,[h_n(s)-h(s)]\,,\, A^{2\alpha}U_n(s) - A^{2\alpha}U_n(\bar{s}_N)\big)\Big|\,ds,$$

$$
\begin{aligned}
I^2_{n,N} &= \int_0^T \Big| \Big( \big[\sigma(s,u(s)) - \sigma(\bar{s}_N,u(s))\big]\,[h_n(s)-h(s)]\,,\, A^{2\alpha}U_n(\bar{s}_N)\Big)\Big|\, ds,\\
I^3_{n,N} &= \int_0^T \Big| \Big( \big[\sigma(\bar{s}_N,u(s)) - \sigma(\bar{s}_N,u(\bar{s}_N))\big]\,[h_n(s)-h(s)]\,,\, A^{2\alpha}U_n(\bar{s}_N)\Big)\Big|\, ds,\\
I^4_{n,N} &= \sup_{1\le k\le 2^N}\ \sup_{t_{k-1}\le t\le t_k} \Big| \Big(\sigma(t_k,u(t_k))\int_{t_{k-1}}^t (h_n(s)-h(s))ds\,,\, A^{2\alpha}U_n(t_k)\Big)\Big|,\\
I^5_{n,N} &= \sum_{1\le k\le 2^N} \Big| \Big(\sigma(t_k,u(t_k))\int_{t_{k-1}}^{t_k} [h_n(s)-h(s)]\,ds\,,\, A^{2\alpha}U_n(t_k)\Big)\Big|.
\end{aligned}
$$

The Cauchy-Schwarz inequality, (4.34), (**C5**) and (4.10) imply that for some constants $C_i$, $i = 0,\cdots,4$, which depend on $k_0$, $\bar{K}_i$, $\bar{L}_1$, $\tilde{C}$, $M$ and $T$, but do not depend on $n$ and $N$,

$$
\begin{aligned}
I^1_{n,N} &\le C_0\Big(\int_0^T \big(\|u_n(s)-u_n(\bar{s}_N)\|^2 + \|u(s)-u(\bar{s}_N)\|^2\big)ds\Big)^{\frac12}\Big(\int_0^T |h_n(s)-h(s)|_0^2 ds\Big)^{\frac12}\\
&\le C_1\, 2^{-\frac{N}{2}}, \qquad (4.36)\\
I^3_{n,N} &\le C_0\Big(\int_0^T \|u(s)-u(\bar{s}_N)\|^2 ds\Big)^{\frac12} 2\sqrt{M} \le C_3\, 2^{-\frac{N}{2}}, \qquad (4.37)\\
I^4_{n,N} &\le C_0\, 2^{-\frac{N}{2}}\Big(1+\sup_{0\le t\le T}\|u(t)\|\Big)\sup_{0\le t\le T}\big(\|u(t)\|+\|u_n(t)\|\big)2\sqrt{M} \le C_4\, 2^{-\frac{N}{2}}. \qquad (4.38)
\end{aligned}
$$

Furthermore, the Hölder regularity of $\sigma(.,u)$ from condition **(C5)** implies that

$$
\begin{aligned}
I^2_{n,N} \le& \bar{C}2^{-N\gamma}\sup_{0\le t\le T}\big(\|u(t)\|+\|u_n(t)\|\big)\\
&\times\int_0^T (1+\|u(s)\|)(|h(s)|_0+|h_n(s)|_0)\,ds \le C_2\, 2^{-N\gamma}. \qquad (4.39)
\end{aligned}
$$

For fixed $N$ and $k = 1,\cdots,2^N$, as $n\to\infty$, the weak convergence of $h_n$ to $h$ implies that of $\int_{t_{k-1}}^{t_k}(h_n(s)-h(s))ds$ to 0 weakly in $H_0$. Since $\sigma(t_k,u(t_k))$ is a compact operator, we deduce that for fixed $k$ the sequence $\sigma(t_k,u(t_k))\int_{t_{k-1}}^{t_k}(h_n(s)-h(s))ds$ converges to 0 strongly in $H$ as $n\to\infty$. Since $\sup_{n,k}\|U_n(t_k)\|\le 2\sqrt{\tilde{C}}$, we have $\lim_n I^5_{n,N}=0$. Thus (4.35)–(4.39) yield for every integer $N\ge 1$

$$
\limsup_{n\to\infty}\sup_{t\le T}\|U_n(t)\|^2_\alpha \le C2^{-N(\frac12\wedge\gamma)}.
$$

Since $N$ is arbitrary, we deduce that $\sup_{0\le t\le T}\|U_n(t)\|_\alpha\to 0$ as $n\to\infty$. This shows that every sequence in $K_M$ has a convergent subsequence. Hence $K_M$ is a sequentially relatively compact subset of $\mathscr{X}$. Finally, let $\{u_n\}$ be a sequence of elements of $K_M$ which converges to $v$ in $\mathscr{X}$. The above argument shows that there exists a subsequence $\{u_{n_k}, k\ge 1\}$ which converges to some element $u_h\in K_M$ for the uniform topology on $\mathscr{C}([0,T],V)$ endowed with the $\|\cdot\|_\alpha$ norm. Hence $v=u_h$, $K_M$ is a closed subset of $\mathscr{X}$, and this completes the proof of the proposition. □

**Proof of Theorem 4.2:** Propositions 4.5 and 4.4 imply that the family $\{u^\nu\}$ satisfies the Laplace principle, which is equivalent to the large deviation principle, in $\mathscr{X}$ defined in (4.7) with the rate function defined by (4.8); see Theorem 4.4 in [4] or Theorem 5 in [5]. This concludes the proof of Theorem 4.2. $\square$

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