

## SIGN PATTERNS THAT ALLOW A POSITIVE OR NONNEGATIVE LEFT INVERSE\*

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**Abstract.** An  $m$  by  $n$  sign pattern  $\mathcal{S}$  is an  $m$  by  $n$  matrix with entries in  $\{+, -, 0\}$ . Such a sign pattern allows a positive (resp., nonnegative) left inverse, provided that there exist an  $m$  by  $n$  matrix  $A$  with the sign pattern  $\mathcal{S}$  and an  $n$  by  $m$  matrix  $B$  with only positive (resp., nonnegative) entries satisfying  $BA = I_n$ , where  $I_n$  is the  $n$  by  $n$  identity matrix. For  $m > n \geq 2$ , a characterization of  $m$  by  $n$  sign patterns with no rows of zeros that allow a positive left inverse is given. This leads to a characterization of all  $m$  by  $n$  sign patterns with  $m \geq n \geq 2$  that allow a positive left inverse, giving a generalization of the known result for the square case, which involves a related bipartite digraph. For  $m \geq n$ ,  $m$  by  $n$  sign patterns with all entries in  $\{+, 0\}$  and  $m$  by 2 sign patterns with  $m \geq 2$  that allow a nonnegative left inverse are characterized, and some necessary or sufficient conditions for a general  $m$  by  $n$  sign pattern to allow a nonnegative left inverse are presented.

**Key words.** bipartite digraph, nonnegative left inverse, positive left inverse, positive left null-vector, sign pattern, strong Hall

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**1. Introduction.** An  $m$  by  $n$  sign pattern  $\mathcal{S} = [s_{ij}]$  is an  $m$  by  $n$  matrix with entries in  $\{+, -, 0\}$ . If a sign pattern  $\mathcal{S}$  has all entries in  $\{+, 0\}$ , then  $\mathcal{S}$  is a *nonnegative* sign pattern. A *subpattern* of  $\mathcal{S}$  is an  $m$  by  $n$  sign pattern  $\mathcal{U} = [u_{ij}]$  such that  $u_{ij} = 0$  whenever  $s_{ij} = 0$ . If  $\mathcal{U}$  is a subpattern of  $\mathcal{S}$ , then  $\mathcal{S}$  is a *superpattern* of  $\mathcal{U}$ . The *sign pattern class*  $Q(\mathcal{S})$  of an  $m$  by  $n$  sign pattern  $\mathcal{S}$  is the set of  $m$  by  $n$  matrices  $A = [a_{ij}]$  such that  $\text{sgn}(a_{ij}) = s_{ij}$  for all  $i, j$ . If  $A \in Q(\mathcal{S})$ , then  $A$  is a *realization* of  $\mathcal{S}$ .

Let  $A = [a_{ij}]$  be an  $m$  by  $n$  matrix. If each entry of  $A$  is positive (resp., nonnegative), then  $A$  is *positive* (resp., *nonnegative*), written  $A > 0$  (resp.,  $A \geq 0$ ). A *left inverse* of an  $m$  by  $n$  matrix  $A$  is an  $n$  by  $m$  matrix  $B$  such that  $BA = I_n$ , where  $I_n$  denotes the  $n$  by  $n$  identity matrix. If  $B > 0$ , then  $B$  is a *positive* left inverse (abbreviated as *PLI*) of  $A$ . If  $B \geq 0$ , then  $B$  is a *nonnegative* left inverse (abbreviated as *NLI*) of  $A$ . In general, neither a PLI nor an NLI of  $A$  is unique. It is easily verified that  $A$  has a left inverse if and only if  $\text{rank } A = n$ ; thus, if  $A$  has a left inverse, then necessarily  $m \geq n$ . An  $m$  by  $n$  sign pattern  $\mathcal{S}$  *allows a positive (resp., nonnegative) left inverse*, provided there exist  $A \in Q(\mathcal{S})$  and a matrix  $B > 0$  (resp.,  $B \geq 0$ ) such that  $BA = I_n$ . Note that if  $P_1$  and  $P_2$  are permutation matrices, then  $\mathcal{S}$  allows a PLI (resp., an NLI) if and only if  $P_1\mathcal{S}P_2$  allows a PLI (resp., an NLI).

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A motivation for studying PLIs and NLIs comes from determining the qualitative behavior of solutions of  $A^T x = b$  with  $A$  an  $m$  by  $n$  matrix; see, for example, [2, Chapter 1] and [5] for applications in economics. Specifically,  $A$  has a PLI (resp., an NLI) if and only if for each  $n$  by 1 nonzero vector  $b \geq 0$  there exists an  $m$  by 1 vector  $x > 0$  (resp.,  $x \geq 0$ ) satisfying  $A^T x = b$  or equivalently  $x^T A = b^T$ ; see Proposition 4.1 for a proof.

Square sign patterns with entries in  $\{+, -\}$  that allow a positive (left) inverse are characterized in [6], and this characterization is extended to arbitrary square sign patterns in [4]. These results are summarized in [2, section 9.2]. In section 2, we characterize nonsquare sign patterns that allow a PLI, and combine the square and nonsquare characterizations. In section 3, we discuss sign patterns that allow an NLI. More specifically, we characterize nonnegative sign patterns and  $m$  by 2 sign patterns with  $m \geq 2$  that allow an NLI, and present some necessary or sufficient conditions for general  $m$  by  $n$  sign patterns with  $m \geq n$  to allow an NLI. We conclude with some remarks in section 4.

**2. Positive left inverses.** We begin this section with a necessary and sufficient condition for a column sign pattern to allow a PLI or an NLI.

PROPOSITION 2.1. *Let  $\mathcal{S} = (s_1, s_2, \dots, s_m)^T$  be an  $m$  by 1 sign pattern. Then the following are equivalent:*

- (i)  $\mathcal{S}$  has a + entry.
- (ii)  $\mathcal{S}$  allows a PLI.
- (iii)  $\mathcal{S}$  allows an NLI.

*Proof.* Suppose there is an index  $i \in \{1, 2, \dots, m\}$  with  $s_i = +$ . For  $j \in \{1, \dots, m\}$ , set

$$a_j = \begin{cases} 1 & \text{if } j \neq i \text{ and } s_j = +, \\ -1 & \text{if } j \neq i \text{ and } s_j = -, \\ 0 & \text{if } j \neq i \text{ and } s_j = 0, \\ 1 + \sum_{k \neq i} |a_k| & \text{if } j = i. \end{cases}$$

Then  $A = (a_1, \dots, a_m)^T \in Q(\mathcal{S})$ , and  $(1, 1, \dots, 1)A = 1 + \sum_{k \neq i} (|a_k| + a_k) = c > 0$ . This implies that  $\frac{1}{c}(1, 1, \dots, 1)$  is a PLI of  $A$ . Thus,  $\mathcal{S}$  allows a PLI.

It is clear that (ii) implies (iii). Next, suppose that the sign pattern  $\mathcal{S}$  allows an NLI. Then there exist  $A = (a_1, \dots, a_m)^T \in Q(\mathcal{S})$  and  $B = (b_1, \dots, b_m) \geq 0$  such that  $BA = 1$ , i.e.,  $\sum_{j=1}^m b_j a_j = 1 > 0$ . This implies that there exists an  $i$  with  $b_i a_i > 0$ . Since  $b_i \geq 0$ , it follows that  $b_i > 0$ ; hence  $a_i > 0$  and thus  $s_i = +$ .  $\square$

We now consider  $m \geq n \geq 2$ . The following two lemmas give necessary conditions for a sign pattern to allow a PLI.

LEMMA 2.2. *Let  $\mathcal{S}$  be an  $m$  by  $n$  sign pattern with  $n \geq 2$ . If  $\mathcal{S}$  allows a PLI, then each column of  $\mathcal{S}$  has a + and a - entry.*

*Proof.* Suppose that there exist  $A \in Q(\mathcal{S})$  and an  $n$  by  $m$  positive matrix  $B$  such that  $BA = I_n$ . Let  $i \in \{1, 2, \dots, n\}$ . Since the  $(i, i)$ -entry of  $BA$  is 1 and each entry of  $B$  is positive, it follows that some entry in column  $i$  of  $A$  is positive. Hence, column  $i$  of  $\mathcal{S}$  has a + entry.

Since  $n \geq 2$ , there exists  $j \in \{1, \dots, n\}$  with  $j \neq i$ . The  $(j, i)$ -entry of  $BA$  is 0, so since  $B > 0$  and at least one entry of column  $i$  of  $A$  is positive, it follows that at least one entry of column  $i$  of  $A$  must be negative. Thus, column  $i$  of  $\mathcal{S}$  has a - entry.  $\square$

An  $m$  by  $n$  sign pattern  $\mathcal{S}$  with  $n \geq 2$  is *strong Hall*, provided that for every nonempty proper subset  $\gamma$  of  $\{1, 2, \dots, n\}$  the submatrix of  $\mathcal{S}$  consisting of the columns

indexed by  $\gamma$  has nonzero entries in at least  $|\gamma| + 1$  rows (see [3]). Note that if  $\mathcal{S}$  is strong Hall, then necessarily  $m \geq n$ . Also, for  $m \geq n$ ,  $\mathcal{S}$  is not strong Hall if and only if there exist permutation matrices  $P_1$  and  $P_2$  such that

$$(2.1) \quad P_1 \mathcal{S} P_2 = \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ O & \mathcal{S}_{22} \end{bmatrix},$$

where  $\mathcal{S}_{11}$  is a  $k$  by  $\ell$  sign pattern for some integers  $k, \ell$  with  $n > \ell \geq 1$  and  $k \leq \ell$ .

LEMMA 2.3. *Let  $\mathcal{S}$  be an  $m$  by  $n$  sign pattern with  $n \geq 2$ . If  $\mathcal{S}$  allows a PLI, then  $\mathcal{S}$  is strong Hall.*

*Proof.* To prove the contrapositive, assume that  $\mathcal{S}$  is not strong Hall. If  $m < n$ , then it is clear that  $\mathcal{S}$  does not allow a PLI. Otherwise, without loss of generality, we may assume that  $\mathcal{S}$  has the form (2.1). If  $k < \ell$ , then the first  $\ell$  columns of each realization of  $\mathcal{S}$  are linearly dependent, and hence  $\mathcal{S}$  does not allow a PLI.

Otherwise,  $k = \ell < n$ . Suppose that there exists a matrix  $A = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix} \in Q(\mathcal{S})$  with a left inverse  $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ , where  $B_{11}$  is an  $\ell$  by  $\ell$  matrix. Clearly, the  $\ell$  by  $\ell$  matrix  $A_{11}$  is invertible, and by  $BA = I_n$ , it follows that  $\begin{bmatrix} B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} \\ O \end{bmatrix} = O$ . Thus,  $B_{21}A_{11} = O$ , and since  $A_{11}$  is invertible, the  $(n - \ell)$  by  $\ell$  matrix  $B_{21} = O$ . Since  $n - \ell \geq 1$  and  $\ell \geq 1$ , every left inverse of a matrix in  $Q(\mathcal{S})$  has a zero entry, and hence  $\mathcal{S}$  does not allow a PLI.  $\square$

Note that if  $\mathcal{S}$  is a square sign pattern of order  $n \geq 2$ , then  $\mathcal{S}$  is strong Hall if and only if  $\mathcal{S}$  is fully indecomposable (see [3]), and  $\mathcal{S}$  allows a PLI if and only if  $\mathcal{S}$  allows a positive inverse. The next theorem, first proved in [4], provides a characterization of square sign patterns that allow a positive inverse. In order to recall this characterization, we use the following definition as in [1] and [2]. Let  $\mathcal{S} = [s_{ij}]$  be an  $m$  by  $n$  sign pattern. The *bipartite digraph*  $D(\mathcal{S})$  of  $\mathcal{S}$  is the digraph with row vertices  $u_1, \dots, u_m$ , column vertices  $v_1, \dots, v_n$ , an arc  $u_i \rightarrow v_j$  if  $s_{ij} = +$ , and an arc  $v_j \rightarrow u_i$  if  $s_{ij} = -$ . Note that there exists at most one arc between  $u_i$  and  $v_j$ .

THEOREM 2.4 (see [2, Theorem 9.2.1]). *An  $n$  by  $n$  square sign pattern  $\mathcal{S}$  with  $n \geq 2$  allows a positive (left) inverse if and only if  $\mathcal{S}$  is strong Hall and the bipartite digraph  $D(\mathcal{S})$  of  $\mathcal{S}$  is strongly connected.*

Let  $\mathcal{S}$  be an  $m$  by  $n$  sign pattern and let  $D(\mathcal{S})$  be its bipartite digraph. A *strong component* of  $D(\mathcal{S})$  is a maximal strongly connected subdigraph of  $D(\mathcal{S})$ . If  $\alpha$  is a strong component of  $D(\mathcal{S})$ , then  $|\alpha|$  denotes the number of vertices in  $\alpha$ .

Remark 2.5. Let  $\alpha$  be a strong component of  $D(\mathcal{S})$ . Since  $D(\mathcal{S})$  is a bipartite digraph with no cycles of length 2, it follows that if  $|\alpha| \geq 2$ , then  $\alpha$  has at least two row vertices and at least two column vertices.

Let  $\alpha_1, \alpha_2, \dots, \alpha_t$  be the strong components of  $D(\mathcal{S})$ . The *condensed* digraph  $CD(\mathcal{S})$  of  $\mathcal{S}$  has vertices  $\alpha_1, \alpha_2, \dots, \alpha_t$  and an arc  $\alpha_i \rightarrow \alpha_j$  if and only if  $i \neq j$  and  $D(\mathcal{S})$  has at least one arc from a vertex in  $\alpha_i$  to a vertex in  $\alpha_j$ . A strong component  $\alpha_i$  of  $D(\mathcal{S})$  is a *source* if there is no arc coming into  $\alpha_i$  in  $CD(\mathcal{S})$  and there is at least one arc coming out of  $\alpha_i$  in  $CD(\mathcal{S})$ ;  $\alpha_i$  is a *sink* if there is no arc coming out of  $\alpha_i$  in  $CD(\mathcal{S})$  and there is at least one arc coming into  $\alpha_i$  in  $CD(\mathcal{S})$ ; and  $\alpha_i$  is *isolated* if there are no arcs coming into or out of  $\alpha_i$  in  $CD(\mathcal{S})$ .

LEMMA 2.6. *Let  $\mathcal{S}$  be an  $m$  by  $n$  sign pattern which has a  $+$  and a  $-$  entry in each column and no rows of zeros. Then the following hold for  $D(\mathcal{S})$ :*

- (i) *Each sink and source strong component of  $D(\mathcal{S})$  has at least one row vertex.*
- (ii) *Each isolated strong component has at least two row vertices.*

*Proof.* (i) Let  $\alpha$  be a sink or source strong component. If  $|\alpha| = 1$ , then since each column of  $\mathcal{S}$  has a  $+$  and a  $-$  entry, it follows that no sink or source strong component

consists of exactly one column vertex. Hence,  $\alpha$  is a row vertex. If  $|\alpha| \geq 2$ , then Remark 2.5 implies that  $\alpha$  has at least one row vertex.

(ii) By the assumptions on the rows and columns of  $\mathcal{S}$ , there is no isolated strong component with exactly one vertex. Hence, by Remark 2.5, each isolated strong component has at least two row vertices.  $\square$

Let  $A$  be an  $m$  by  $n$  matrix with  $m \geq n$ . If there exists an  $m$  by 1 vector  $y > 0$  satisfying  $y^T A = 0$ , then  $y^T$  is a *positive left nullvector* of  $A$ . The following theorem gives a characterization of nonsquare sign patterns with no rows of zeros that allow a PLI. Note that conditions for such a sign pattern to allow a PLI are weaker than those for square sign patterns (Theorem 2.4), although the bipartite digraph is used in our proof for a nonsquare sign pattern.

**THEOREM 2.7.** *For  $m > n \geq 2$ , let  $\mathcal{S}$  be an  $m$  by  $n$  sign pattern with no rows of zeros. Then the following are equivalent:*

- (i) *There exists a matrix  $A \in Q(\mathcal{S})$  with a PLI and a positive left nullvector.*
- (ii)  *$\mathcal{S}$  allows a PLI.*
- (iii) *Each column of  $\mathcal{S}$  has a + and a - entry, and  $\mathcal{S}$  is strong Hall.*

*Proof.* Clearly, (i) implies (ii). By Lemmas 2.2 and 2.3, (ii) implies (iii).

To prove that (iii) implies (i), assume that  $\mathcal{S}$  is strong Hall and that  $\mathcal{S}$  has a + and a - entry in each column. We claim that it suffices to show that there exists an  $m$  by  $(m - n)$  sign pattern  $\mathcal{C}$  so that the  $m$  by  $m$  sign pattern  $[\mathcal{S} \mid \mathcal{C}]$  allows a positive (left) inverse. To prove this claim, suppose there exists an  $m$  by  $m$  matrix  $[A \mid C] \in Q([\mathcal{S} \mid \mathcal{C}])$  with a positive (left) inverse  $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$  where  $B_1$  is an  $n$  by  $m$  positive matrix and  $B_2$  is an  $(m - n)$  by  $m$  positive matrix. Then  $B_1 A = I_n$  and hence  $B_1$  is a PLI of  $A$ , implying that  $\mathcal{S}$  allows a PLI. In addition, since  $B_2 A = O$  and  $B_2$  has at least one positive row,  $A$  has a positive left nullvector. Therefore, by Theorem 2.4, it suffices to find an  $m$  by  $(m - n)$  sign pattern  $\mathcal{C}$  such that the  $m$  by  $m$  sign pattern  $[\mathcal{S} \mid \mathcal{C}]$  is strong Hall and its bipartite digraph  $D([\mathcal{S} \mid \mathcal{C}])$  is strongly connected.

Consider the bipartite digraph  $D(\mathcal{S})$  of  $\mathcal{S}$ . Let  $\alpha_1, \alpha_2, \dots, \alpha_t$  be its strong components, where  $\alpha_1, \dots, \alpha_k$  are sinks,  $\alpha_{k+1}, \dots, \alpha_{k+\ell}$  are sources,  $\alpha_{k+\ell+1}, \dots, \alpha_{k+\ell+r}$  are isolated, and  $\alpha_{k+\ell+r+1}, \dots, \alpha_t$  are neither sinks, sources, nor isolated strong components. By Lemma 2.6 (i), each sink and source strong component has a row vertex. Let  $r_i$  be a fixed row vertex of  $\alpha_i$  for each  $i \in \{1, \dots, k + \ell\}$ . Also, by Lemma 2.6 (ii), each isolated strong component has at least two row vertices. Let  $r_i^+, r_i^-$  be distinct fixed row vertices of  $\alpha_i$  for each  $i \in \{k + \ell + 1, \dots, k + \ell + r\}$ . Let  $\mathcal{C}_{n+1}$  be the  $m$  by 1 column sign pattern with nonzero  $j$ th coordinate:

$$(2.2) \quad \begin{cases} + & \text{if } u_j \in \{r_1, \dots, r_k\} \cup \{r_{k+\ell+1}^-, \dots, r_{k+\ell+r}^-\}, \\ - & \text{if } u_j \in \{r_{k+1}, \dots, r_{k+\ell}\} \cup \{r_{k+\ell+1}^+, \dots, r_{k+\ell+r}^+\}, \\ + & \text{otherwise.} \end{cases}$$

Then  $D([\mathcal{S} \mid \mathcal{C}_{n+1}])$  is obtained from  $D(\mathcal{S})$  by appending a new column vertex  $c_{n+1}$ , and arcs  $r_j \rightarrow c_{n+1}$  if  $r_j$  is in a sink component;  $c_{n+1} \rightarrow r_j$  if  $r_j$  is in a source component;  $r_j^- \rightarrow c_{n+1}$  and  $c_{n+1} \rightarrow r_j^+$  if  $r_j^-$  and  $r_j^+$  are in the same isolated component; as well as some additional arcs coming into vertex  $c_{n+1}$ .

To prove that  $D([\mathcal{S} \mid \mathcal{C}_{n+1}])$  is strongly connected, we show that for each vertex  $w$  of  $D(\mathcal{S})$  there exists in  $D([\mathcal{S} \mid \mathcal{C}_{n+1}])$  a walk from  $c_{n+1}$  to  $w$  and a walk from  $w$  to  $c_{n+1}$ . Note that if  $w$  is not in an isolated strong component of  $D(\mathcal{S})$ , then there is a walk from  $w$  to a vertex in a sink strong component  $\alpha_i$  of  $D(\mathcal{S})$  ( $i \in \{1, \dots, k\}$ ). Since  $\alpha_i$  is strongly connected, this walk from  $w$  can be extended to the fixed row vertex  $r_i$  of  $\alpha_i$ . By (2.2), there is an arc  $r_i \rightarrow c_{n+1}$  in  $D([\mathcal{S} \mid \mathcal{C}_{n+1}])$ . Hence, there is

a walk from  $w$  to  $c_{n+1}$ . Similarly, there is a walk from  $c_{n+1}$  to  $w$ .

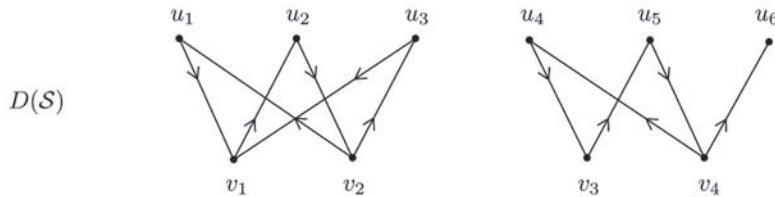
Next, suppose that  $w$  is a vertex in an isolated strong component  $\alpha_i$  in  $D(\mathcal{S})$  ( $i \in \{k + \ell + 1, \dots, k + \ell + r\}$ ). Since  $\alpha_i$  is strongly connected, there is a walk from  $w$  to the fixed row vertex  $r_i^-$  of  $\alpha_i$ . By (2.2), there are arcs  $r_i^- \rightarrow c_{n+1}$  and  $c_{n+1} \rightarrow r_i^+$  in  $D([\mathcal{S} \mid \mathcal{C}_{n+1}])$ . Since  $\alpha_i$  is strongly connected, there is a walk from  $r_i^+$  to  $w$ . Thus, there exist a walk from  $w$  to  $c_{n+1}$  and a walk from  $c_{n+1}$  to  $w$ .

Finally, define  $\mathcal{C}_{n+2}, \dots, \mathcal{C}_m$  to be  $m$  by 1 column sign patterns, each having no zeros, at least one +, and at least one - entry. Then it is easily verified that  $D([\mathcal{S} \mid \mathcal{C}_{n+1} \mid \dots \mid \mathcal{C}_m])$  is strongly connected. Since  $\mathcal{S}$  is strong Hall and  $[\mathcal{C}_{n+1} \mid \dots \mid \mathcal{C}_m]$  has no zeros, it is clear that  $[\mathcal{S} \mid \mathcal{C}_{n+1} \mid \dots \mid \mathcal{C}_m]$  is strong Hall, completing the proof.  $\square$

*Example 2.8.* Consider the 6 by 4 sign pattern

$$\mathcal{S} = \begin{bmatrix} + & - & 0 & 0 \\ - & + & 0 & 0 \\ + & - & 0 & 0 \\ 0 & 0 & + & - \\ 0 & 0 & - & + \\ 0 & 0 & 0 & - \end{bmatrix}$$

with



Each column of  $\mathcal{S}$  has a + and a - entry, and  $\mathcal{S}$  is strong Hall. Thus, by Theorem 2.7,  $\mathcal{S}$  allows a PLI. However,  $D(\mathcal{S})$  is not strongly connected, illustrating a distinction between the nonsquare and square cases (see Theorem 2.4). In fact,  $D(\mathcal{S})$  has one sink strong component  $\alpha_1$  that consists of vertex  $u_6$ , one source strong component  $\alpha_2$  with vertices  $u_4, u_5, v_3, v_4$ , and one isolated strong component  $\alpha_3$  with vertices  $u_1, u_2, u_3, v_1, v_2$ . Taking  $r_1 = u_6$ ,  $r_2 = u_5$ ,  $r_3^+ = u_1$ , and  $r_3^- = u_2$  in the proof of Theorem 2.7, it follows that

$$\mathcal{C}_5 = \begin{bmatrix} - \\ + \\ + \\ + \\ - \\ + \end{bmatrix}.$$

The last column  $\mathcal{C}_6$  can be taken to be any 6 by 1 column having a + and a - entry, and no zeros. Let  $\mathcal{C} = [\mathcal{C}_5 \mid \mathcal{C}_6]$ . In order to determine a matrix  $[A \mid \mathcal{C}] \in Q([\mathcal{S} \mid \mathcal{C}])$  with a positive (left) inverse  $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ , the algorithm described in the proof of [2, Theorem 9.2.1] can be used. Then  $B_1$  is a PLI of  $A$ , and the rows of  $B_2$  are positive left nullvectors of  $A$ .

The next lemma is used to prove Theorem 2.10, in which square and nonsquare cases are combined.

LEMMA 2.9. *Let  $\mathcal{S}$  be an  $m$  by  $n$  sign pattern with  $n \geq 2$  and let  $\mathcal{T}$  be the sign pattern obtained from  $\mathcal{S}$  by deleting the rows of zeros in  $\mathcal{S}$ . Then*

- (i)  $\mathcal{S}$  is strong Hall if and only if  $\mathcal{T}$  is strong Hall, and
- (ii)  $\mathcal{S}$  allows a positive (nonnegative) left inverse if and only if  $\mathcal{T}$  allows a positive (nonnegative) left inverse.

*Proof.* Without loss of generality, assume that  $\mathcal{S} = \begin{bmatrix} \mathcal{T} \\ \mathcal{O} \end{bmatrix}$ . The proof of (i) follows immediately from the definition of strong Hall.

To prove (ii), suppose first that  $\mathcal{S}$  allows a PLI. Let  $A_1 \in Q(\mathcal{T})$  and  $A = \begin{bmatrix} A_1 \\ \mathcal{O} \end{bmatrix} \in Q(\mathcal{S})$  have  $B = [B_1 \ B_2]$  as a PLI. Then  $B_1 A_1 = I_n$  and hence  $\mathcal{T}$  allows a PLI. Next, suppose that  $\mathcal{T}$  allows a PLI. Let  $A_1 \in Q(\mathcal{T})$  have  $B_1$  as a PLI. With  $J$  denoting the all 1's matrix, it is easily verified that  $B = [B_1 \ J]$  is a PLI for  $A = \begin{bmatrix} A_1 \\ \mathcal{O} \end{bmatrix} \in Q(\mathcal{S})$ . Hence,  $\mathcal{S}$  allows a PLI. The nonnegative case can be shown by a similar argument to that above.  $\square$

THEOREM 2.10. *Let  $m \geq n \geq 2$ . The  $m$  by  $n$  sign pattern  $\mathcal{S}$  allows a PLI if and only if*

- (i) each column of  $\mathcal{S}$  has a + and a - entry;
- (ii)  $\mathcal{S}$  is strong Hall; and
- (iii) the bipartite digraph  $D(\mathcal{S}_1)$  of  $\mathcal{S}_1$  is strongly connected whenever  $\mathcal{S}$  is permutationally equivalent to  $\begin{bmatrix} \mathcal{S}_1 \\ \mathcal{O} \end{bmatrix}$  and  $\mathcal{S}_1$  is an  $n$  by  $n$  sign pattern.

*Proof.* For the necessity, suppose that  $\mathcal{S}$  allows a PLI. Then (i) and (ii) follow from Lemmas 2.2 and 2.3, and (iii) follows from Theorem 2.4 and Lemma 2.9 (ii).

For the sufficiency, first assume  $m = n$ . Then  $\mathcal{S}$  is permutationally equivalent to  $\mathcal{S}_1$ , and by Theorem 2.4 the result follows from (ii) and (iii). Next, suppose that  $m > n$ . If  $\mathcal{S}$  has no rows of zeros, then, by Theorem 2.7, the result follows from (i) and (ii). Otherwise, without loss of generality, assume that  $\mathcal{S} = \begin{bmatrix} \mathcal{T} \\ \mathcal{O} \end{bmatrix}$ , where  $\mathcal{T}$  has no rows of zeros. By Lemma 2.9 (i), it follows from (ii) that  $\mathcal{T}$  is strong Hall. Thus, if  $\mathcal{T}$  is an  $n$  by  $n$  sign pattern, then (iii) and Theorem 2.4 imply that  $\mathcal{T}$  allows a PLI. By Lemma 2.9 (ii), this implies that  $\mathcal{S}$  allows a PLI. Otherwise, since it follows from (i) that each column of  $\mathcal{T}$  has a + and a - entry, Theorem 2.7 implies that  $\mathcal{T}$  allows a PLI. Therefore, by Lemma 2.9 (ii),  $\mathcal{S}$  allows a PLI.  $\square$

Remark 2.11. For  $m \geq n \geq 2$ , let  $\mathcal{S}$  be an  $m$  by  $n$  sign pattern. Then the following hold:

- (i) If  $\mathcal{S}$  satisfies (i), (ii), and (iii) in Theorem 2.10, then so does every superpattern of  $\mathcal{S}$ . Hence, if  $\mathcal{S}$  allows a PLI, then every superpattern of  $\mathcal{S}$  allows a PLI.
- (ii) Suppose that  $\mathcal{S} = \begin{bmatrix} \mathcal{S}_1 \\ \mathcal{O} \end{bmatrix}$ , where  $\mathcal{S}_1$  is a square sign pattern, satisfies (iii) in Theorem 2.10. Then, in contrast with Theorem 2.7 (i), there is no matrix  $A = \begin{bmatrix} A_1 \\ \mathcal{O} \end{bmatrix} \in Q(\mathcal{S})$  with a PLI that also has a positive left nullvector, since the equation  $[y^T \ z^T] \begin{bmatrix} A_1 \\ \mathcal{O} \end{bmatrix} = 0$  and the fact that  $A_1$  is nonsingular together imply that  $y = 0$ .

The following theorem gives sufficient conditions for an  $m$  by  $n$  sign pattern with  $m > n \geq 1$  to have a realization with a PLI and a positive left nullvector.

THEOREM 2.12. *Let  $\mathcal{S}$  be an  $m$  by  $n$  sign pattern with  $m > n$  and let  $\mathcal{T}$  be the  $t$  by  $n$  sign pattern obtained from  $\mathcal{S}$  by deleting the rows of zeros in  $\mathcal{S}$ .*

- (i) *If  $n = 1$  and  $\mathcal{T}$  has a + and a - entry, then there exists a matrix in  $Q(\mathcal{S})$  with a PLI and a positive left nullvector.*
- (ii) *If  $t > n \geq 2$  and  $\mathcal{T}$  allows a PLI, then there exists a matrix in  $Q(\mathcal{S})$  with a PLI and a positive left nullvector.*

*Proof.* (i) By Proposition 2.1, a + entry implies the existence of  $A \in Q(\mathcal{S})$  with a

PLI. Since  $A$  has a positive and a negative entry, it can be easily verified that  $A$  has a positive left nullvector.

(ii) When  $m = t$ , the result follows by Theorem 2.7. If  $m > t > n$ , then Theorem 2.7 implies that there exists a matrix  $A \in Q(\mathcal{T})$  with a PLI  $B$  and a positive left nullvector  $y^T$ . Note that the positive matrix  $[B \mid J]$  is a PLI and the vector  $[y^T \ 1 \cdots 1]$  is a positive left nullvector of the matrix  $\begin{bmatrix} A \\ O \end{bmatrix} \in Q(\mathcal{S})$ . Hence, the result follows.  $\square$

**3. Nonnegative left inverses.** In this section we determine structures of nonsquare sign patterns that allow an NLI, as well as structures of NLIs.

For  $m \geq n$ , let  $\mathcal{S}$  be an  $m$  by  $n$  sign pattern with a realization of rank  $n$ . Then, by induction, it can be shown that  $\mathcal{S}$  is permutationally equivalent to

$$(3.1) \quad \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} & \cdots & \mathcal{S}_{1k} \\ O & \mathcal{S}_{22} & \cdots & \mathcal{S}_{2k} \\ \vdots & & \ddots & \vdots \\ O & \cdots & O & \mathcal{S}_{kk} \end{bmatrix},$$

where  $k \geq 1$ ,  $\mathcal{S}_{ii}$  is a square fully indecomposable sign pattern for each  $i \in \{1, \dots, k-1\}$ , and  $\mathcal{S}_{kk}$  is strong Hall. Note that  $\mathcal{S}$  is strong Hall if and only if  $k = 1$ . If  $\mathcal{S}$  is an  $n$  by  $n$  fully indecomposable sign pattern, then  $\mathcal{S}$  allows a nonnegative (left) inverse if and only if  $\mathcal{S}$  allows a positive (left) inverse; see [2, Theorems 9.2.1 and 9.2.3]. In addition, [2, Theorem 9.2.6] provides a complete characterization of  $n$  by  $n$  partly decomposable sign patterns that allow a nonnegative (left) inverse.

*Remark 3.1.* Suppose  $m > n$ . Let  $\mathcal{S}'$  be the square submatrix of  $\mathcal{S}$  obtained by deleting the columns and rows associated with  $\mathcal{S}_{kk}$ . Suppose that  $\mathcal{S}$  allows an NLI. Then the square sign pattern  $\mathcal{S}'$  also allows an NLI. Hence, for  $k = 2$ ,  $\mathcal{S}'$  is fully indecomposable and must satisfy one of the equivalent conditions in [2, Theorem 9.2.1] (see also Theorem 2.4), and for  $k \geq 3$ ,  $\mathcal{S}'$  is partly decomposable and must satisfy the conditions in [2, Theorem 9.2.6]. Furthermore, by an argument similar to that in the proof of Lemma 2.3, it is easily verified that an NLI  $B$  of a matrix in  $Q(\mathcal{S})$  has the block form  $B = [B_{ij}]$  with  $1 \leq i, j \leq k$  and the  $(i, j)$ -block  $B_{ij} = O$  whenever  $i > j$ . Thus, it follows that the strong Hall sign pattern  $\mathcal{S}_{kk}$  also allows an NLI.

We now investigate various necessary and/or sufficient conditions for a strong Hall nonsquare sign pattern to allow an NLI. We first consider strong Hall sign patterns with a + and a - entry in each column.

**PROPOSITION 3.2.** *For  $m > n \geq 2$ , let  $\mathcal{S}$  be an  $m$  by  $n$  strong Hall sign pattern with a + and a - entry in each column, and let  $\mathcal{T}$  be the  $t$  by  $n$  sign pattern obtained from  $\mathcal{S}$  by deleting the rows of zeros in  $\mathcal{S}$ . If  $t > n$ , then  $\mathcal{S}$  allows an NLI. If  $t = n$ , then  $\mathcal{S}$  allows an NLI if and only if  $D(\mathcal{T})$  is strongly connected.*

*Proof.* The result follows directly from Theorem 2.10 and the fact that if  $\mathcal{S}$  allows a PLL, then  $\mathcal{S}$  allows an NLI.  $\square$

Let  $\mathcal{I}_n$  denote the  $n$  by  $n$  sign pattern with  $I_n$  as a realization, i.e.,  $I_n \in Q(\mathcal{I}_n)$ . Clearly,  $\mathcal{I}_n$  allows an NLI. Thus, in order to allow an NLI, an  $m$  by  $n$  sign pattern with  $m \geq n$  need not have a - entry in each column as is required to allow a PLI (see Lemma 2.2), but clearly must have a + entry in each column. We first consider the case that  $\mathcal{S}$  has a nonnegative column having only + or 0 entries. For ease of notation, we sometimes use  $(M)_{ij}$  to denote the  $(i, j)$ -entry of a matrix  $M$ .

**PROPOSITION 3.3.** *For  $m \geq n \geq 2$ , let  $\mathcal{S}$  be an  $m$  by  $n$  sign pattern with at least one nonnegative column. If  $\mathcal{S}$  allows an NLI, then each nonnegative column has at most  $m - n + 1$  positive entries.*

*Proof.* Let  $B$  be an NLI of  $A \in Q(\mathcal{S})$ , and let  $t$  be the number of positive entries in any nonnegative column of  $A$ . Without loss of generality, assume that the first column of  $A$  is a nonnegative column with its first  $t$  entries positive. Since  $(BA)_{h1} = 0$  for each  $h \in \{2, \dots, n\}$ , it follows that  $B$  has the block form  $B = [B_{ij}]$  with  $1 \leq i, j \leq 2$ , where the  $(2, 1)$ -block  $B_{21}$  is the  $(n - 1)$  by  $t$  zero matrix. Hence, the equality  $\text{rank } B = n$  implies that the rank of the  $(n - 1)$  by  $(m - t)$  matrix  $B_{22}$  is  $n - 1$ . Thus,  $n - 1 \leq m - t$  and the result follows.  $\square$

If all columns are nonnegative, then the following result gives a necessary and sufficient condition for such a sign pattern to allow an NLI.

**THEOREM 3.4.** *For  $m \geq n \geq 1$ , let  $\mathcal{S}$  be an  $m$  by  $n$  nonnegative sign pattern. Then  $\mathcal{S}$  allows an NLI if and only if  $\mathcal{S}$  is permutationally equivalent to*

$$\begin{bmatrix} \mathcal{I}_n \\ \mathcal{T} \end{bmatrix},$$

where  $\mathcal{T}$  is an  $(m - n)$  by  $n$  nonnegative sign pattern.

*Proof.* The case  $n = 1$  follows directly from Proposition 2.1. Suppose that  $n \geq 2$ .

For the sufficiency, assume without loss of generality that

$$\mathcal{S} = \begin{bmatrix} \mathcal{I}_n \\ \mathcal{T} \end{bmatrix}.$$

Let  $T \in Q(\mathcal{T})$  and  $A = \begin{bmatrix} \mathcal{I}_n \\ T \end{bmatrix} \in Q(\mathcal{S})$ . Since  $[I_n \mid O]A = I_n$ , it follows that  $\mathcal{S}$  allows an NLI.

For the necessity, suppose that  $\mathcal{S} = [s_{ij}]$  allows an NLI; i.e., there exist  $A = [a_{ij}] \in Q(\mathcal{S})$  and an  $n$  by  $m$  nonnegative matrix  $B = [b_{ij}]$  such that  $BA = I_n$ . Let  $i \in \{1, \dots, n\}$ . Since  $(BA)_{ii} = 1$ , there exists  $j_i \in \{1, \dots, m\}$  such that  $b_{ij_i} a_{j_i i} > 0$ . This implies that  $s_{j_i i} = +$ . Also, for each  $k \in \{1, \dots, n\} \setminus \{i\}$ ,  $(BA)_{ik} = 0$  implies that  $b_{ij_i} a_{j_i k} = 0$ . Thus, row  $j_i$  of  $\mathcal{S}$  is equal to row  $i$  of  $\mathcal{I}_n$ . As this holds for each  $i \in \{1, \dots, n\}$ , the result follows.  $\square$

*Remark 3.5.* Let  $\mathcal{S} = \begin{bmatrix} \mathcal{I}_n \\ \mathcal{J} \end{bmatrix}$  be the  $m$  by  $n$  nonnegative sign pattern with  $m \geq n \geq 2$ , where  $\mathcal{J}$  is the sign pattern with all entries positive. Then, by Theorem 3.4,  $\mathcal{S}$  allows an NLI. However, in contrast with Remark 2.11 (i), Theorem 3.4 implies that no nonnegative superpattern of  $\mathcal{S}$  (except  $\mathcal{S}$  itself) allows an NLI.

Next, we consider sign patterns that have at least one nonnegative column and at least one column with a + and a - entry. We use  $e_i$  to denote the  $i$ th column vector of an identity matrix.

**THEOREM 3.6.** *For  $m \geq n \geq 2$ , let  $\mathcal{S}$  be an  $m$  by  $n$  sign pattern that has  $p \geq 1$  nonnegative columns and  $n - p \geq 1$  columns with a + and a - entry. Suppose that  $\mathcal{S}$  allows an NLI. Then  $\mathcal{S}$  is permutationally equivalent to a matrix of the form*

$$(3.2) \quad \begin{bmatrix} \mathcal{I}_p & \mathcal{S}_{12} \\ \mathcal{S}_{21} & \mathcal{S}_{22} \\ O & \mathcal{S}_{32} \end{bmatrix},$$

where  $\mathcal{S}_{21}$  is an  $r$  by  $p$  nonnegative sign pattern with no rows of zeros,  $O$  is an  $s$  by  $p$  zero matrix with  $s \geq 1$ , and each of the last  $n - p$  columns of  $\mathcal{S}$  has a + and a - entry. Furthermore, if  $\mathcal{S}$  is strong Hall, then  $\mathcal{S}_{21}$  is not vacuous and has no column of zeros.

*Proof.* Without loss of generality, we may assume that the first  $p$  columns of  $\mathcal{S}$  are nonnegative, and that each of the last  $n - p$  columns of  $\mathcal{S}$  has a + and a - entry.



Since  $\mathcal{S}$  allows an NLI, so does the  $m$  by  $p$  nonnegative sign pattern consisting of the first  $p$  columns of  $S$ . Therefore, by Theorem 3.4, we may permute the rows of  $\mathcal{S}$  to obtain a matrix of the form (3.2), where  $S_{21}$  is a nonnegative matrix with no row of zeros,  $O$  is an  $s$  by  $p$  zero matrix with  $s \geq 0$ , and each of the last  $n - p$  columns has a  $+$  and a  $-$  entry.

Let  $A$  be a matrix in  $Q(\mathcal{S})$  that has an NLI, say  $B$ . Since  $BA = I_n$ , each of the vectors  $e_1^T, \dots, e_n^T$  is a nontrivial, nonnegative linear combination of the rows of  $A$ . Since the first  $p$  columns of  $A$  are nonnegative and  $n > p$ , this requires that  $s \geq 1$ , and we conclude that  $\mathcal{S}$  has the desired form.

If  $S_{21}$  is vacuous or has a column of zeros, then  $\mathcal{S}$  has an  $(m - 1)$  by 1 zero submatrix. Hence  $\mathcal{S}$  is not strong Hall, and the result follows by taking the contrapositive.  $\square$

PROPOSITION 3.7. *For  $m \geq n \geq 2$ , let  $\mathcal{S}$  be an  $m$  by  $n$  sign pattern that has  $p \geq 1$  nonnegative columns and  $n - p \geq 1$  columns with a  $+$  and a  $-$  entry. Suppose that  $\mathcal{S}$  allows an NLI and has the form (3.2). Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ O & A_{32} \end{bmatrix} \in Q(\mathcal{S})$$

have an NLI  $B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$ , where each of  $A_{21}, A_{22}, B_{12}$ , and  $B_{22}$  may be vacuous if  $\mathcal{S}$  is not strong Hall. Then the following hold:

- (i)  $B_{11}$  is a diagonal matrix, and  $B_{21}$  and  $B_{22}$  are zero matrices.
- (ii) The sign pattern  $\mathcal{S}_{32}$  allows an NLI.
- (iii) If row  $q$  of  $S_{21}$  has at least two positive entries, then column  $q$  of  $B_{12}$  is a zero column.
- (iv) Each column of  $B_{12}$  has at most one positive entry. Furthermore, the sign pattern of  $B_{12}$  is a subpattern of  $S_{21}^T$ .

*Proof.* Assume that  $S_{21}$  is not vacuous.

Since  $BA = I_n$ , it follows that  $B_{21}A_{11} + B_{22}A_{21} = O$ . Moreover, since  $B_{21}, B_{22}, A_{11}$ , and  $A_{21}$  are nonnegative, and no row of  $A_{11}$  or  $A_{21}$  is all zeros,  $B_{21} = O$  and  $B_{22} = O$ . Also,  $BA = I_n$  implies that  $B_{11}A_{11} + B_{12}A_{21} = I_p$ . Since  $B_{11}, B_{12}, A_{11}$ , and  $A_{21}$  are nonnegative, both  $B_{11}A_{11}$  and  $B_{12}A_{21}$  are diagonal matrices. Since  $A_{11} \in Q(\mathcal{I}_p)$ ,  $A_{11}$  is an invertible diagonal matrix, and hence  $B_{11}$  is a diagonal matrix. Thus, (i) is proven.

Since  $B_{21}$  and  $B_{22}$  are zero matrices, and  $BA = I_n$ ,  $B_{23}$  is an NLI of  $A_{32}$ , and (ii) is proven.

Since  $B_{12}A_{21}$  is a diagonal matrix and  $B_{12}$  is nonnegative, the  $i$ th row of  $B_{12}A_{21}$  is a nonnegative linear combination of the rows of  $A_{21}$  (weighted by the entries of the  $i$ th row of  $B_{12}$ ). As the  $i$ th row of  $B_{12}A_{21}$  is a nonnegative multiple of  $e_i^T$ , and  $A_{21}$  is a nonnegative matrix with no row of zeros, it follows that if the  $(i, j)$ -entry of  $B_{12}$  is nonzero, then the  $j$ th row of  $A_{21}$  is a multiple of  $e_i^T$ . In particular, this implies that each column of  $B_{12}$  has at most one nonzero entry. If the  $j$ th row of  $A_{21}$  has at least two positive entries, then column  $j$  of  $B_{12}$  is a column of zeros, proving (iii). If the  $(i, j)$ -entry of  $B_{12}$  is nonzero, then the  $(j, i)$ -entry of  $A_{21}$  is nonzero, completing the proof of (iv).

If  $S_{21}$  is vacuous, then  $A_{21}, A_{22}, B_{12}$ , and  $B_{22}$  are vacuous, in which case the proofs of (i) for  $B_{11}, B_{21}$  and (ii) are similar, but statements (i) for  $B_{22}$ , (iii), and (iv) are vacuous.  $\square$

For  $m \geq 2$ , Proposition 3.2, Theorem 3.4, and the following theorem completely characterize the  $m$  by 2 sign patterns that allow an NLI.

**THEOREM 3.8.** *For  $m \geq 2$ , let  $\mathcal{S}$  be an  $m$  by 2 sign pattern such that the first column is nonnegative and the second column has a + and a - entry. Then  $\mathcal{S}$  allows an NLI if and only if the first column of  $\mathcal{S}$  has a + entry and  $[0 +]$  is a row of  $\mathcal{S}$ .*

*Proof.* Suppose that  $\mathcal{S}$  allows an NLI. Then the first column of  $\mathcal{S}$  also allows an NLI. Hence, Theorem 3.4 implies that the first column of  $\mathcal{S}$  has a + entry. By Theorem 3.6, we may assume without loss of generality that  $\mathcal{S}$  is of the form (3.2). Since  $\mathcal{S}_{32}$  is a column sign pattern, Propositions 3.7 (ii) and 2.1 imply that  $\mathcal{S}_{32}$  has a + entry. Hence,  $[0 +]$  is a row of  $\mathcal{S}$ .

For the converse, suppose that the first column of  $\mathcal{S}$  has a + entry and  $[0 +]$  is a row of  $\mathcal{S}$ . Suppose that  $[+ -]$  is also a row of  $\mathcal{S}$ . Then without loss of generality,  $A \in \mathcal{S}$  has the form

$$\begin{bmatrix} a & -b \\ u & v \\ 0 & c \end{bmatrix},$$

where  $a, b, c > 0$ , and  $u$  and  $v$  are  $(m - 2)$  by 1 vectors. It is easy to verify that

$$\begin{bmatrix} 1/a & O & b/ac \\ 0 & O & 1/c \end{bmatrix}$$

is an NLI of  $A$ .

Next suppose that  $[+ -]$  is not a row of  $\mathcal{S}$ . Then without loss of generality,  $A \in \mathcal{S}$  has the form

$$\begin{bmatrix} a & b \\ u & v \\ 0 & -c \\ 0 & d \end{bmatrix},$$

where  $a, c, d > 0$ ,  $b \geq 0$ , and  $u$  and  $v$  are  $(m - 3)$  by 1 vectors. It is easy to verify that

$$\begin{bmatrix} 1/a & O & b/ac & 0 \\ 0 & O & 1/c & 2/d \end{bmatrix}$$

is an NLI of  $A$ .

Hence,  $\mathcal{S}$  allows an NLI.  $\square$

Note that the proof of Theorem 3.8 actually shows that if  $\mathcal{S}$  is an  $m$  by 2 matrix whose first column is nonnegative, second column has a + and a - entry, and  $[0 +]$  is one of its rows, then *each* matrix with sign pattern  $\mathcal{S}$  has an NLI.

*Example 3.9.* The strong Hall sign pattern

$$\mathcal{S} = \begin{bmatrix} + & - \\ + & - \\ 0 & + \end{bmatrix}$$

does not allow a PLI (by Lemma 2.2), but does allow an NLI (by Theorem 3.8) since

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 2 \end{bmatrix} = I_2.$$

In general (as noted in the introduction) an NLI is not unique. For instance,

$$\begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$$

is another NLI of the above matrix.

In the next theorem, it is shown that if a sign pattern  $\mathcal{S}$  of the form (3.2) has a  $(3, 2)$ -block  $\mathcal{S}_{32}$  that allows an NLI or PLI, then some conditions on the negative entries in  $\mathcal{S}_{12}$  insure that  $\mathcal{S}$  allows an NLI.

**THEOREM 3.10.** *For  $m \geq n \geq 2$ , let  $\mathcal{S}$  be an  $m$  by  $n$  sign pattern of the form (3.2) with  $p \geq 1$ ,  $n - p \geq 1$ , and  $\mathcal{S}_{21}$ ,  $\mathcal{S}_{22}$  arbitrary. Then the following hold:*

- (i) *If  $\mathcal{S}_{32}$  allows an NLI and  $\mathcal{S}_{12}$  has only 0 or  $-$  entries, then  $\mathcal{S}$  allows an NLI.*
- (ii) *If  $\mathcal{S}_{32}$  allows a PLI and each row of  $\mathcal{S}_{12}$  has a  $-$  entry, then  $\mathcal{S}$  allows an NLI.*

*Proof.* (i) Let

$$(3.3) \quad A = \begin{bmatrix} I_p & A_{12} \\ A_{21} & A_{22} \\ O & A_{32} \end{bmatrix} \in Q(\mathcal{S}),$$

where  $-A_{12} \geq 0$  and  $A_{32}$  has  $B_{23}$  as an NLI. Let

$$(3.4) \quad B = \begin{bmatrix} I_p & O & B_{13} \\ O & O & B_{23} \end{bmatrix}$$

with  $B_{13} = -A_{12}B_{23}$ , which is a nonnegative matrix. Then  $B \geq 0$ ,  $BA = I_n$ , and hence the result follows.

(ii) Let  $A \in Q(\mathcal{S})$  be of the form (3.3) and let  $B$  be of the form (3.4). If  $B_{23}$  is a PLI of  $A_{32}$  and  $B_{13} = -A_{12}B_{23}$ , then  $B_{13} > 0$ , provided that the negative entries of  $A_{12}$  are sufficiently large in magnitude, and  $BA = I_n$  as required.  $\square$

**4. Concluding remarks.** In section 3, we have characterized nonnegative sign patterns, strong Hall sign patterns with each column having a  $+$  and a  $-$  entry, and  $m$  by 2 sign patterns that allow an NLI. For other cases, we have given some necessary or sufficient conditions for  $\mathcal{S}$  to allow an NLI. A characterization for the blocks of the last column of a sign pattern  $\mathcal{S}$  of the form (3.1) with  $k \geq 2$  that allows an NLI remains open. We conclude by showing (in Theorem 4.2) that some conditions on the submatrix  $\mathcal{S}_{kk}$  of a sign pattern  $\mathcal{S}$  of the form (3.1) with  $k \geq 2$  insure that  $\mathcal{S}$  allows an NLI for arbitrary  $\mathcal{S}_{1k}, \dots, \mathcal{S}_{k-1,k}$ .

Let  $\mathcal{S}$  allow a PLI and  $A \in Q(\mathcal{S})$ . The following proposition, which is used to prove Theorem 4.2, describes a relation between a PLI of  $A$  and the qualitative behavior of solutions of  $x^T A = b^T$ . The latter equation is given in the introduction as motivation for studying PLIs and NLIs.

**PROPOSITION 4.1.** *For  $m \geq n$ , let  $A$  be an  $m$  by  $n$  matrix. Then  $A$  has a PLI if and only if for each  $n$  by 1 nonzero vector  $b \geq 0$  there exists an  $m$  by 1 vector  $x > 0$  satisfying  $x^T A = b^T$ .*

*Proof.* Suppose that an  $n$  by  $m$  matrix  $B > 0$  is a PLI of  $A$ . For an  $n$  by 1 nonzero vector  $b \geq 0$ , it is clear that  $(b^T B)A = b^T$  and  $b^T B > 0$ . Hence, the result follows.

Next, suppose that for each  $n$  by 1 nonzero vector  $b \geq 0$  there exists an  $m$  by 1 vector  $x > 0$  satisfying  $x^T A = b^T$ . Take  $b$  to be the  $i$ th column  $e_i$  of  $I_n$  and let  $x_i > 0$

be a solution of  $x^T A = e_i^T$ . Then the matrix

$$B = \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix}$$

is a PLI of  $A$ .  $\square$

**THEOREM 4.2.** For  $m > s \geq 1$ ,  $n > t \geq 1$ , and  $m > n$ , let  $\mathcal{S}_{11}$  be an  $s$  by  $t$  sign pattern that allows an NLI and let  $\mathcal{S}_{22}$  be an  $(m-s)$  by  $(n-t)$  sign pattern that allows a PLI. Suppose that if  $n-t=1$ , then  $\mathcal{S}_{22}$  has a  $-$  entry, and if  $n-t \geq 2$ , then  $\mathcal{S}_{22}$  is not permutationally equivalent to the sign pattern  $\begin{bmatrix} \mathcal{T} \\ \mathcal{O} \end{bmatrix}$  in which  $\mathcal{T}$  is a square sign pattern. Then, for an arbitrary  $s$  by  $(n-t)$  sign pattern  $\mathcal{S}_{12}$ , the sign pattern  $\mathcal{S} = \begin{bmatrix} \mathcal{S}_{11} & \mathcal{S}_{12} \\ \mathcal{O} & \mathcal{S}_{22} \end{bmatrix}$  allows an NLI.

*Proof.* Let  $A_{11}$  be a matrix in  $Q(\mathcal{S}_{11})$  with  $B_{11}$  as an NLI. By Theorem 2.12, there exists  $A_{22} \in Q(\mathcal{S}_{22})$  that has a PLI  $B_{22}$  and a positive left nullvector  $y^T$ . Let  $A_{12} \in Q(\mathcal{S}_{12})$ . Then  $A_{12}$  can be written as  $A_{12} = V_1 - V_2$ , where  $V_1, V_2 \geq 0$  and the entrywise (Hadamard) product  $V_1 \circ V_2 = O$ . Let  $v_i^T \geq 0$  for  $1 \leq i \leq s$  denote row  $i$  of  $V_1$ . If  $v_i \neq 0$ , then by Proposition 4.1 there exists an  $(m-s)$  by 1 vector  $x_i > 0$  such that  $x_i^T A_{22} = v_i^T$ . If  $v_i = 0$ , then  $x_i^T A_{22} = v_i^T = 0$  when  $x_i^T = y^T$ . Thus,  $K_1 = [x_1, \dots, x_s]^T > 0$  and  $K_1 A_{22} = V_1$ . Similarly, there exists  $K_2 > 0$  such that  $K_2 A_{22} = V_2$ .

Let  $A_{12}(\epsilon) = \epsilon V_1 - V_2 = (\epsilon K_1 - K_2) A_{22}$  for a sufficiently small  $\epsilon > 0$  such that  $K_2 - \epsilon K_1 > 0$ . Note that  $V_1 \circ V_2 = O$  implies that  $A_{12}(\epsilon) \in Q(\mathcal{S}_{12})$ . Let  $B_{12} = B_{11}(K_2 - \epsilon K_1)$ . Since  $K_2 - \epsilon K_1 > 0$  and  $B_{11} \geq 0$  with no rows of zeros, it follows that  $B_{12} > 0$ . It can be easily verified that  $\begin{bmatrix} B_{11} & B_{12} \\ \mathcal{O} & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12}(\epsilon) \\ \mathcal{O} & A_{22} \end{bmatrix} = I_n$ . Hence, the result follows.  $\square$

**Remark 4.3.** Take  $\mathcal{S}_{11}$  and  $\mathcal{S}_{22}$  in Theorem 4.2 to be  $\mathcal{S}'$  in Remark 3.1 and  $\mathcal{S}_{kk}$  in the form (3.1) with  $k \geq 2$ , respectively. Then the conditions on  $\mathcal{S}_{kk}$  in Theorem 4.2 insure that the sign pattern  $\mathcal{S}$  of the form (3.1) with  $k \geq 2$  allows an NLI for arbitrary  $\mathcal{S}_{1k}, \dots, \mathcal{S}_{k-1,k}$ .

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