Math 2200

Calculus 1 Lecture Notes

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Introduction

These are lecture notes for Calculus 1, adapted from the Spring semester of 2024.

Some of these notes utilize content from the open-access textbook:

> Herman, E., Strang, G. (2020). *Calculus Volume 1*. OpenStax. https://openstax.org/details/books/calculus-volume-1.

The format and some content is based on a document originally developed by Regene M. DePiero and has gone through several edits but there are most likely still typos lingering here and there. Be mindful of this as you read. Any errors/comments/questions can be sent to cwisema3@uwyo.edu.

Contents

List of Figures

List of Tables

Lecture 1

Introduction to Limits

Sections
2.2.1-2.2.5

Introduction to Limits

“The limit does not exist!" This famous line was uttered by Cady (played by Lindsay Lohan) in the movie, Mean Girls. And in a rare occurrence for Hollywood math, not only was she right, but her reasoning was also correct. But what is a limit and why do we care about them? The full answer to this question won’t show up for a few lectures, but as an introduction, consider the following problem originally posed by the ancient Greek philosopher Zeno.

You’re late to class and running down the hall. But before you can get to the classroom, you must cover half the distance between where you started and the classroom. And after you’ve covered this distance, you must cover half of the remaining distance (i.e. three quarters of the total distance). And after this, you still have another half of the distance remaining. And you can keep dividing up the distance this way, infinitely. So how is it that you ever make it to class if you have an infinite number of things to do in a finite time? Or even worse, how did you ever get started? Before you ran half the distance, you had to run half of that (i.e. one quarter of the full distance), etc. Our experience tells us that we are able to move from point A to point B, but where is the error in Zeno’s logic? To solve this problem, we need limits. We’ll begin with an example.

Consider the graph of the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

pictured in Figure 1.1. We see that $f(x)$ has a “hole” at $x = 2$. In other words, $f(x)$ is defined for every value of x except $x = 2$.

Even though $f(x)$ is not defined at $x = 2$, we can still determine its behavior as we get closer and closer to this x value. We do this by looking at the behavior on the left hand side and on the right hand side of $x = 2$.

In Table 1.1 we see some values of $f(x)$ for x values close to $x = 2$.

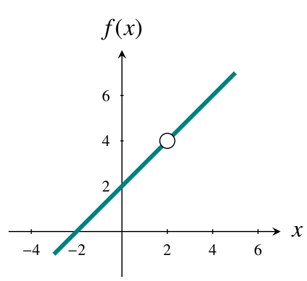


Fig. 1.1: Graph of $f(x) = \frac{x^2-4}{x-2}$

x approaches 2 from the left → ← x approaches 2 from the right

x	1.9	1.99	1.999	2	2.0001	2.001	2.01
$f(x)$	3.9	3.99	3.999	?	4.0001	4.001	4.01

$f(x)$ approaches 4 → ← $f(x)$ approaches 4

Table 1.1: Some values of $f(x) = \frac{x^2-4}{x-2}$

We observe that $f(x)$ approaches 4 as x approaches 2 from *both* the left and the right hand sides. The value that $f(x)$ approaches is know as the **limit** of $f(x)$ as x approaches 2.

Definition 1.1: Limit of a Function

Let $f(x)$ be defined on an open interval about a (except possibly at a itself). If $f(x)$ gets arbitrarily close to finite L for all x sufficiently close to a (but not equal to a), we write

$$\lim_{x\to a} f(x) = L$$

and say that the limit of $f(x)$ as x approaches a is equal to L.

The terms “arbitrarily" and “sufficiently" in the above definition have a specified mathematical meaning. We will not discuss the formal definitions in this course, but you can learn these in section 2.5 of the textbook.

Example 1.1: Evaluating a Limit Numerically

Evaluate the function $f(x) = \frac{x}{\sqrt{x+1}-1}$ at several points near $x = 0$ and use the result to estimate

$$\lim_{x\to 0} \frac{x}{\sqrt{x+1}-1}$$

Solution. We list our values in a table to more easily examine the behavior.

x approaches 0 from the left → ← x approaches 0 from the right

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99950	1.999950	?	2.00005	2.00050	2.00499

$f(x)$ approaches 2 → ← $f(x)$ approaches 2

From the results in the table we estimate that

$$\lim_{x\to 0} \frac{x}{\sqrt{x+1}-1} = 2$$

We can also see this behavior by examining the graph of $f(x)$

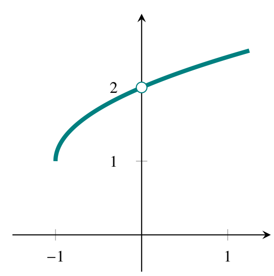


Fig. 1.2: Graph of $f(x) = \frac{x}{\sqrt{x+1}-1}$

Limits that Do Not Exist

There are several situations where limits do not exist. We will demonstrate these through some examples. Before we begin, we need an important definition. We have mentioned $\lim_{x\to a} f(x) = L$. This is what is known as a **two sided limit** since we examine behavior as we approach a from the left *and* from the right according to its graph. It is also possible to talk about these different “sides” of the limit as limits themselves.

Definition 1.2: One Sided Limits

Suppose that f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to a value L as x approaches a from the left then the **left side limit** is written as

$$\lim_{x\to a^-} f(x) = L$$

Suppose that f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to a value L as x approaches a from the right then the **right side limit** is written as

$$\lim_{x\to a^+} f(x) = L$$

Example 1.2: $f(x)$ has Differing Left and Right Side Limits

Determine whether $\lim_{x\to 0} \frac{|x|}{x}$ exists.

Solution. Let $f(x) = \frac{|x|}{x}$. Since $f(x)$ contains the absolute value we must examine two cases; when $x > 0$ and when $x < 0$.

- When $x > 0$ we have

$$\frac{|x|}{x} = \frac{x}{x} = 1$$

using limit notation we write

$$\lim_{x\to 0^+} \frac{|x|}{x} = 1$$

- When $x < 0$ we have

$$\frac{|-x|}{-x} = \frac{x}{-x} = -1$$

using limit notation we write

$$\lim_{x\to 0^-} \frac{|x|}{x} = -1$$

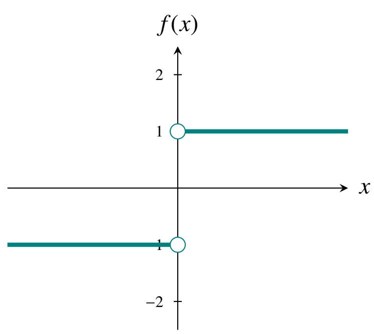


Fig. 1.3: Graph of $f(x) = \frac{|x|}{x}$

By examining the graph of $f(x)$ we can see that as we approach the value $x = 0$ our $f(x)$ approaches two different values. Which value do we use for $\lim_{x\to 0} f(x)$? The answer is neither.

Since we have conflicting information for which value $f(x)$ is approaching we say that *the limit does not exist*. This is also most commonly abbreviated by DNE.

Our findings in this example are precisely stated in the following theorem.

Theorem 1.1: Existence of Two Sided Limits

$\lim_{x \to a} f(x) = L$ if and only if (abbreviated "iff")

$$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

In other words, a limit exists iff its left side limit is equal to its right side limit.

⚠ *If and only if* has a specific meaning in math. This is logical implication in both directions. If I slept for 8 hours last night then I slept for 480 minutes last night, **also** if I slept for 480 minutes last night then I slept for 8 hours. This is not the case for every theorem. Some logical implications only go in one direction. If today is Christmas, then it is the 25th of the month. But if it is the 25th of the month, it is not necessarily Christmas.

Example 1.3: $f(x)$ Increases or Decreases without Bound

Determine whether the limit as x goes to 0 exists for the function $f(x) = \frac{1}{x^2}$

Solution. Lets establish a table of function values as x approaches 0.

x approaches 0 from the left → ← x approaches 0 from the right

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	100	10000	1000000	?	1000000	10000	100

$f(x)$ approaches ∞ → ← $f(x)$ approaches ∞

We see that as x gets closer and closer to zero, $f(x)$ gets larger and larger.

In this case, we see say that $f(x)$ **increases without bound**. This means that the limit L approaches infinity, which is not a real number!

Thus, according to our definition this limit *does not exist*. This is also true for functions that **decreases without bound**.

We can also see this in the graph of $f(x) = \frac{1}{x^2}$ in Figure 1.4.

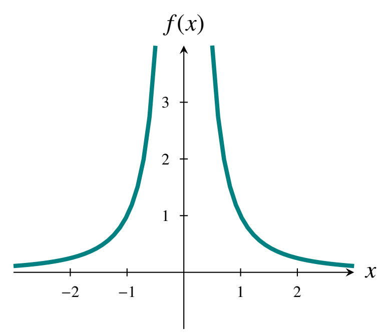


Fig. 1.4: Graph of $f(x) = \frac{1}{x^2}$

We will examine these types of limits in more detail in Lesson 2. Next, we discuss a typical example that often arises when dealing with periodic functions like the trigonometric functions.

Example 1.4: $f(x)$ has Oscillating Behavior

Determine whether $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ exists.

Solution. The graph of $f(x) = \sin\left(\frac{1}{x}\right)$ is seen in Figure 1.5.

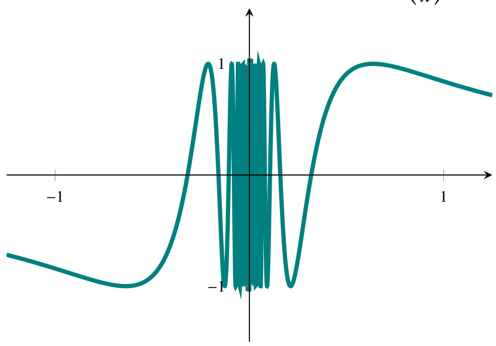


Fig. 1.5: Graph of $f(x) = \sin\left(\frac{1}{x}\right)$

We observe that as x gets closer and closer to zero, $f(x)$ oscillates between -1 and 1 infinitely as often as x approaches 0. So $f(x)$ doesn't approach a fixed value as x approaches 0.

In this case we also have a limit that *does not exist*.

To summarize the results from these examples, a limit does not exist when

- $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$
- $f(x)$ increases or decreases without bound as $x \to a$
- $f(x)$ oscillates between two fixed values as $x \to a$

Remark: It is not always possible (or wise) to find limits numerically using a table. In most cases, we must find limits analytically. The word "analytic" in mathematics essentially means exact. In other words, this involves using algebraic methods to find a solution rather than numerical ones.

Properties of Limits

We have seen that $\lim_{x\to a} f(x)$ does not necessarily depend on the value of f at $x = a$ since often it is not even defined there. However, for functions that are continuous at x we can have

$$\lim_{x\to a} f(x) = f(a)$$

This is known as evaluating a limit by **direct substitution**. We simply plug in a to f and evaluate to get our result. For many limits, we must carry out some algebraic manipulation before we can actually use direct substitution.

Before we can start manipulating limits, we need to know what we can and cannot do with limits. First we have some basic properties.

Theorem 1.2: Basic Properties of Limits

1. $\lim_{x\to a} b = b$ **2.** $\lim_{x\to a} x = a$ **3.** $\lim_{x\to a} x^n = a^n$

Example 1.5: Evaluating Basic Limits

Evaluate the following limits.

a.) $\lim_{x\to 2} 3$

Solution. Since $f(x)$ is always 3 it doesn't matter what x-value we approach. In other words, we have $f(2) = 3$ thus,

$$\lim_{x\to 2} 3 = 3$$

b.) $\lim_{x\to -4} x$

Solution. We know that $f(x) = x$ is a continuous function and that $f(-4) = -4$. Thus,

$$\lim_{x\to -4} x = -4$$

c.) $\lim_{x\to 2} x^2$

Solution. Since $f(x) = x^2$ is a continuous function and we know that $f(2) = (2)^2$. Thus,

$$\lim_{x\to 2} x^2 = 2^2 = 4$$

Now what about when we have multiple terms with several different operations involved in a limit? The next lecture discusses properties that will allow us to handle these cases. They are commonly referred to as the **limit laws**.

Lecture 2

The Limit Laws and Techniques for Evaluating Limits

Section
2.3

Theorem 2.1: The Limit Laws

Let c be a real number, let $m > 0$ and $n > 0$ be integers. Assuming that

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

both exist, then the following hold:

1. Sum $\quad \lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. Difference $\quad \lim_{x\to a}(f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. Constant Multiple $\quad \lim_{x\to a} c(f(x)) = c \lim_{x\to a} f(x)$

4. Product $\quad \lim_{x\to a}(f(x) \cdot g(x)) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$

5. Quotient $\quad \lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \quad$ where $\lim_{x\to a} g(x) \neq 0$

6. Power $\quad \lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$

7. Fractional Power $\quad \lim_{x\to a}(f(x))^{n/m} = \left(\lim_{x\to a} f(x)\right)^{n/m}$, where $f(x) \geq 0$ for x near a if n/m is reduced to lowest terms and m is even.

We now examine some examples applying the limit laws. It may seem tedious to write out every step in these examples but it is important to recognize why direct substitution works in certain cases.

Example 2.1: Applying Limit Laws

Evaluate the following limits.

a.) $\lim_{x\to 5}(2x^4 - 3x + 4)$

Solution.

$$\begin{aligned}
\lim_{x\to5}(2x^4-3x+4) &= \lim_{x\to5}2x^4-\lim_{x\to5}3x+\lim_{x\to5}4 && \text{Apply laws 1 \& 2}\\
&= 2\lim_{x\to5}x^4-3\lim_{x\to5}x+\lim_{x\to5}4 && \text{Apply law 3}\\
&= 2\left(\lim_{x\to5}x\right)^4-3\lim_{x\to5}x+\lim_{x\to5}4 && \text{Apply law 6}\\
&= 2(5)^4-3(5)+4=1239 && \text{Use direct substitution}
\end{aligned}$$

b.) $\displaystyle\lim_{x\to-2}\frac{x^3+2x^2-1}{5-3x}$

Solution.

$$\begin{aligned}
\lim_{x\to-2}\frac{x^3+2x^2-1}{5-3x} &= \frac{\lim_{x\to-2}(x^3+2x^2-1)}{\lim_{x\to-2}(5-3x)} && \text{Law 5}\\
&= \frac{\lim_{x\to-2}x^3+\lim_{x\to-2}2x^2-\lim_{x\to-2}1}{\lim_{x\to-2}5-\lim_{x\to-2}3x} && \text{Laws 1 \& 2}\\
&= \frac{\lim_{x\to-2}x^3+2\lim_{x\to-2}x^2-\lim_{x\to-2}1}{\lim_{x\to-2}5-3\lim_{x\to-2}x} && \text{Law 3}\\
&= \frac{\left(\lim_{x\to-2}x\right)^3+2\left(\lim_{x\to-2}x\right)^2-\lim_{x\to-2}1}{\lim_{x\to-2}5-3\lim_{x\to-2}x} && \text{Law 6}\\
&= \frac{(-2)^3+2(-2)^2-1}{5-3(-2)}=-\frac{1}{11} && \text{Direct substitution}
\end{aligned}$$

By applying the limit laws in the previous examples we have demonstrated another valuable theorem for finding limits of polynomial and rational functions. This theorem essentially tells us when we can actually use direct substitution in a limit evaluation.

Theorem 2.2: Limits of Polynomial and Rational Functions

Assume that p and q are polynomials and a is constant.

a. Polynomial Functions: $\displaystyle\lim_{x\to a}p(x)=p(a)$

b. Rational Functions: $\displaystyle\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{p(a)}{q(a)}$ provided $q(a)\neq0$

The limit laws given are also true for one sided limits, with the exception of fractional powers. This rule can be re-stated as follows

Theorem 2.3: One Sided Limits with Fractional Powers

Assume that $m > 0$, $n > 0$ are integers and that n/m is reduced to lowest terms. Then the following hold.

a. $\lim_{x\to a^+} (f(x))^{n/m} = \left(\lim_{x\to a^+} f(x)\right)^{n/m}$, where $f(x) \geq 0$ for x near a with $x > a$ if m is even.

b. $\lim_{x\to a^-} (f(x))^{n/m} = \left(\lim_{x\to a^-} f(x)\right)^{n/m}$, where $f(x) \geq 0$ for x near a with $x < a$ if m is even.

Example 2.2: Finding One-Sided Limits Analytically

Consider the function

$$f(x) = \begin{cases} x^2 + 1, & x < 1 \\ (x-2)^2, & x \geq 1 \end{cases}$$

Find the following.

a.) $\lim_{x\to 1^-} f(x)$

Solution. For $x \to 1^-$ we are approaching 1 from the left hand side, where $x < 1$ so we examine $f(x) = x^2 + 1$. We have

$$\lim_{x\to 1^-} x^2 + 1 = (1)^2 + 1 = 2$$

b.) $\lim_{x\to 1^+} f(x)$

Solution. For $x \to 1^+$ we are approaching 1 from the right hand side, where $x \geq 1$ so we examine $f(x) = (x-2)^2$. We have

$$\lim_{x\to 1^+} (x-2)^2 = (1-2)^2 = (-1)^2 = 1$$

c.) $\lim_{x\to 1} f(x)$

Solution. Since

$$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

then $\lim_{x\to 1} f(x)$ *does not exist* (DNE)

Techniques for Finding Limits

There are two main methods used to algebraically manipulate a limit in order to reach an analytic solution.

Factor and Cancel

This method is most often used when given a rational function. Simply factor the numerator or denominator and then cancel common terms.

Example 2.3: Finding a Limit by Factoring and Canceling

Find $\lim_{x\to -3} \dfrac{x^2+x-6}{x+3}$

Solution. We cannot use direct substitution here since it would result in a zero for the denominator.

$$\begin{aligned}
\lim_{x\to -3} \frac{x^2+x-6}{x+3} &= \lim_{x\to -3} \frac{(x+3)(x-2)}{x+3} && \text{Factor the numerator}\\
&= \lim_{x\to -3} \frac{\cancel{(x+3)}(x-2)}{\cancel{x+3}} && \text{Cancel common terms}\\
&= \lim_{x\to -3} x-2 \\
&= -3-2 = -5 && \text{Use direct substitution}
\end{aligned}$$

Multiplying by the Conjugate

Recall that when we rationalize a denominator (or a numerator) of an algebraic expression we multiply by it's **conjugate**. For example, the expression $a+b$ has the conjugate $a-b$. Multiplying conjugates together gives us

$$(a+b)(a-b) = a^2 - ab + ab - b^2 = a^2 - b^2$$

You should recognize this as the **difference of squares**. We most commonly use this technique when dealing with radicals. The difference of squares property allows us to eliminate the radical. This frees up terms so that they can (potentially) be eliminated.

Remark: In previous math courses you may have been required to "rationalize the denominator" when you have a result like $\frac{2}{\sqrt{2}}$. There is *no* mathematical necessity in writing results in this form! Usually it is actually more convenient to leave the radical in the denominator!

Example 2.4: Finding Limits Using Conjugates

Find $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$.

Solution. We cannot use direct substitution here since it would result in the denominator being equal to zero. In order to find this limit we will use the conjugate of $\sqrt{x+1}-1$. This will allow us to free the terms under the radical.

The conjugate of this algebraic expression is $\sqrt{x+1}+1$.

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} &= \lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \cdot \left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) && \text{Multiply by the conjugate} \\ &= \lim_{x\to 0} \frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)}\end{aligned}$$

Remark: You may be tempted to distribute terms at this step. Don't! Remember that our goal is ultimately cancel the x term in the denominator!

$$\begin{aligned} &= \lim_{x\to 0} \frac{(\sqrt{x+1})^2-(1)^2}{x(\sqrt{x+1}+1)} && \text{Apply difference of squares} \\ &= \lim_{x\to 0} \frac{(x+1)-1}{x(\sqrt{x+1}+1)} \\ &= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{x+1}+1)} && \text{Cancel common terms} \\ &= \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} \\ &= \frac{1}{\sqrt{(0)+1}+1} = \frac{1}{2} && \text{Use direct substitution}\end{aligned}$$

Limits of Trigonometric Functions

The following are some helpful limits to recognize for common transcendental functions.

Theorem 2.4: Limits of Transcendental Functions

Let a be a real number in the domain of the given function.

1. $\lim_{x \to a} \sin x = \sin a$

2. $\lim_{x \to a} \cos x = \cos a$

3. $\lim_{x \to a} \tan x = \tan a$

4. $\lim_{x \to a} b^x = b^a$

5. $\lim_{x \to a} \csc x = \csc a$

6. $\lim_{x \to a} \sec x = \sec a$

7. $\lim_{x \to a} \cot x = \cot a$

8. $\lim_{x \to a} \ln x = \ln a$

The following are some special limits that can be helpful. These results are found by applying the **squeeze theorem** which is covered in the next section.

Theorem 2.5: Special Limits

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$ **2.** $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$ **3.** $\lim_{x \to 0} (1 + x)^{1/x} = e$

The Squeeze Theorem

We have now developed some techniques for finding limits analytically. Unfortunately, much of the time there are limits that we cannot find using these direct methods. The **squeeze theorem** (also sometimes referred to as the sandwich or pinching theorem) is an important tool in finding some of these limits.

Suppose that we have a function $f(x)$ for which we cannot directly find $\lim_{x \to a} f(x)$. In order to use the squeeze theorem we must know two functions that will "squeeze" $f(x)$ between them near the point $x = a$.

Theorem 2.6: The Squeeze Theorem

Suppose we have three functions f, g, and h where

$$g(x) \leq f(x) \leq h(x)$$

If $\lim_{x \to a} g(x) = L = \lim_{x \to a} h(x)$ then,

$$\lim_{x \to a} f(x) = L$$

The best way to see how the squeeze theorem works is to look at an example of how it may be applied.

Example 2.5: Using the Squeeze Theorem

Use the Squeeze Theorem to show that

$$\lim_{x\to 0} x^2\cos(20\pi x) = 0$$

Solution. Let's break this down a bit. With our knowledge of graph transformations we know that $\cos(20\pi x)$ is just $\cos(x)$ with a horizontal stretch. The range remains unchanged. We also know that $\cos(x)$ has range $[-1, 1]$. In other words, $-1 \le \cos(x) \le 1$ and so we have

$$-1 \le \cos(20\pi x) \le 1$$

If we multiply each part of this inequality by x^2 we get

$$-x^2 \le x^2\cos(20\pi x) \le x^2$$

In the middle of this inequality we have our function $f(x)$ for which we want to find the limit.

Our two outside functions which we are "squeezing" $f(x)$ at $x = 0$ are $g(x) = -x^2$ and $h(x) = x^2$. This is seen in Figure 2.1.

Now we have the 3 functions required by the definition.

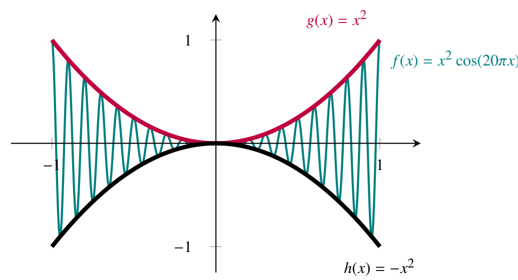


Fig. 2.1: Graph of g, h and f

We know by direct substitution that $\lim_{x\to 0} -x^2 = 0$ and $\lim_{x\to 0} x^2 = 0$. So if we take the limit of each part of this inequality as $x \to 0$ we have

$$\lim_{x\to 0} -x^2 \le \lim_{x\to 0} x^2\cos(20\pi x) \le \lim_{x\to 0} x^2$$

and this gives us that

$$0 \le \lim_{x\to 0} x^2\cos(20\pi x) \le 0$$

Now the only value that can exist between 0 and 0? It's zero! So *by the squeeze theorem*, we must have

$$\lim_{x\to 0} x^2\cos(20\pi x) = 0$$

Lecture 3

Infinite Limits

Sections
2.2.6-2.2.7

Infinite Limits

Consider the function we saw in Example 1.3 in Lesson 1

$$f(x) = \frac{1}{x^2}$$

We see that as x gets closer and closer to zero, $f(x)$ gets larger and larger, in other words, positive infinity.

The limit as x approaches 0 in this case is known as an **infinite limit**.

Technically speaking, infinite limits do not exist but they occur frequently in the study of calculus. Because of this, it is important to learn how to find them and how to interpret what they mean for the behavior of a function.

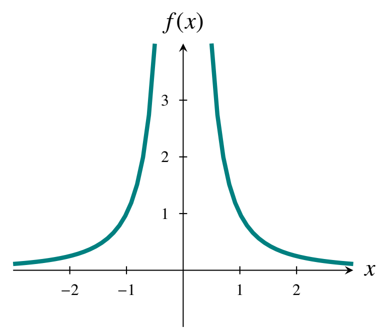


Fig. 3.1: Graph of $f(x) = \frac{1}{x^2}$

Remark: Recall that the term **magnitude** refers to distance from zero, rather than the actual numerical value. This is the same concept as the absolute value function. Thus, we make the distinction *in magnitude* when defining limits that approach negative infinity. This means that $-\infty$ is very large in terms of how far it is from zero, even though it's a "smaller" number in terms of its numerical value. Thinking of this limit as "very small" can lead to confusion between limits that are getting smaller *in magnitude* since these types of limits are actually approaching zero!

⚠ There are many different notations in dealing with infinity. As we will see in greater depth later on in the course, infinity is not a number but rather a concept, and the symbol ∞ is *always* an abbreviation for for this concept. In these notes, we typically make the stylistic choice to use an arrow ($\to$) rather than an equals sign ($=$) when working with infinite limits (this is, in part, to remind us that we can't algebraically manipulate infinity as if it were a number.) In other words, we'll write $\lim_{x\to a} f(x) \to \infty$ rather than $\lim_{x\to a} f(x) = \infty$. Your textbook, as well as many other resources, commonly use an equals sign here and as such, either notation is acceptable.

Definition 3.1: Two-Sided Infinite Limits

Suppose f is defined for all x near a.

- If $f(x)$ grows arbitrarily large for all x sufficiently close, but not equal to a, then

$$\lim_{x\to a} f(x) \to \infty$$

is a **(positive) infinite limit** and we say that the limit of $f(x)$ as x approaches a is infinity.

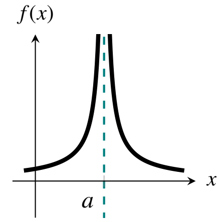


Fig. 3.2: $\lim_{x\to a} f(x) \to \infty$

- If $f(x)$ is *negative* and grows arbitrarily large *in magnitude* for all x sufficiently close, but not equal to a, then

$$\lim_{x\to a} f(x) \to -\infty$$

is a **(negative) infinite limit** and we say that the limit of $f(x)$ as x approaches a is *negative* infinity.

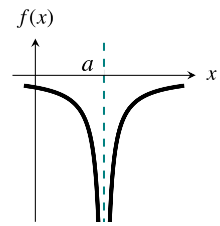


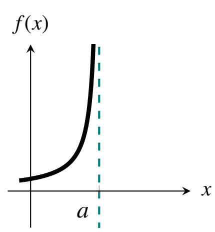
Fig. 3.3: $\lim_{x\to a} f(x) \to -\infty$

In other words, as we get closer and closer to $x = a$ from both the left and the right, our value for $f(x)$ gets larger and larger (or smaller and smaller in the case of $-\infty$). We also have conditions for defining one-sided infinite limits.

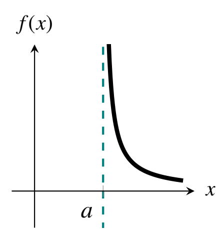
Definition 3.2: One-Sided (Positive) Infinite Limits

Suppose f is defined for all x near a. If $f(x)$ grows arbitrarily large for all x sufficiently close to a (but not equal to a) then

- For $x < a$ we write, $\lim_{x\to a^-} f(x) \to \infty$
- For $x > a$ we write, $\lim_{x\to a^+} f(x) \to \infty$



$\lim_{x\to a^-} f(x) \to \infty$

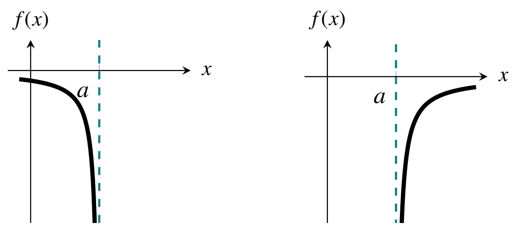


$\lim_{x\to a^+} f(x) \to \infty$

Definition 3.3: One-Sided (Negative) Infinite Limits

If $f(x)$ is *negative* and grows arbitrarily large in *magnitude* for all x sufficiently close to a (but not equal to a) then

- For $x < a$ we write, $\lim_{x \to a^-} f(x) \to -\infty$
- For $x > a$ we write, $\lim_{x \to a^+} f(x) \to -\infty$



$\lim_{x \to a^-} f(x) \to -\infty$ $\lim_{x \to a^+} f(x) \to -\infty$

Vertical Asymptotes

From our study of algebra we known that the vertical line at the point $x = a$ is a **vertical asymptote**. Now we can define these asymptotes by using our understanding of limits.

Definition 3.4: Vertical Asymptotes

The line $x = a$ is called a **vertical asymptote** of a function $f(x)$ if at least one of the following is true.

$$\lim_{x \to a} f(x) \to \infty \qquad \lim_{x \to a^-} f(x) \to \infty \qquad \lim_{x \to a^+} f(x) \to \infty$$

$$\lim_{x \to a} f(x) \to -\infty \qquad \lim_{x \to a^-} f(x) \to -\infty \qquad \lim_{x \to a^+} f(x) \to -\infty$$

Example 3.1: Determining Infinite Limits Graphically

Find all limits of

$$f(x) = \frac{1}{x^2 - 1}$$

Solution. First we re-write $f(x)$ with a factored denominator

$$f(x) = \frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)}$$

The zeros of the denominator are $x = -1$, $x = 1$. These are the vertical asymptotes of $f(x)$. We see the graph of $f(x)$ in Figure 3.4.

We observe the following facts:

- ▶ As $x \to -1$ from the left $f(x) \to \infty$
- ▶ As $x \to -1$ from the right $f(x) \to -\infty$
- ▶ As $x \to 1$ from the left $f(x) \to -\infty$
- ▶ As $x \to 1$ from the right $f(x) \to \infty$

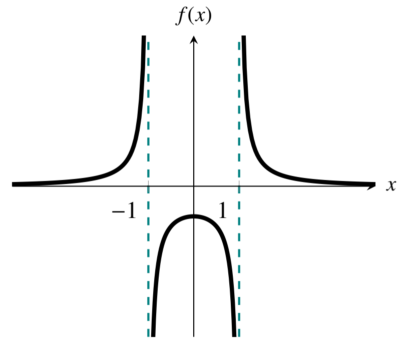


Fig. 3.4: Graph of $f(x) = \frac{1}{x^2-1}$

Thus, our limits are

$$\lim_{x\to -1^-} f(x) \to \infty, \qquad \lim_{x\to -1^+} f(x) \to -\infty, \qquad \lim_{x\to 1^-} f(x) \to -\infty, \qquad \lim_{x\to 1^+} f(x) \to \infty$$

Observe that both $\lim\limits_{x\to -1} f(x)$ and $\lim\limits_{x\to 1} f(x)$ do not exist since the limits on the left and the right are approaching different values.

Finding Infinite Limits Analytically

Our use of the property explained in the introduction is key to helping us deduce whether we have an infinite limit or not.

Example 3.2: Determining Infinite Limits Analytically

Let
$$f(x) = \frac{-4}{x+2}$$
and evaluate the following limits analytically.

a.) $\lim\limits_{x\to -2^-} f(x)$

Solution. For this limit we are approaching $x = -2$ from the left. We investigate what happens to this function for values $x < -2$. Now
$$x < -2 \implies x + 2 < 0$$
This means we have a negative denominator that is getting closer and closer to zero for values of x to the left of -2. Thus, we have a negative number divided by a smaller and smaller negative number. This tells us that our value of $f(x)$ is approaching a larger and larger positive number. Therefore,
$$\lim_{x\to -2^-} \frac{-4}{x+2} \to \infty$$

b.) $\lim\limits_{x\to -2^+} f(x)$

Solution. Now we are approaching $x = -2$ from the right. We investigate what happens to this function for values $x > -2$. Now
$$x > -2 \implies x + 2 > 0$$
This means we have a positive denominator that is getting closer and closer to zero for values of x to the right of -2. Thus, we have a negative number divided by a smaller and smaller positive number. This tells us that our value of $f(x)$ is approaching a larger and larger (in magnitude) negative number. Therefore,
$$\lim_{x\to -2^+} \frac{-4}{x+2} \to -\infty$$

c.) $\lim\limits_{x\to -2} f(x)$

Solution. Since
$$\lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x)$$
the limit as $x \to -2$ does not exist.

Do not be quick to decide that all zeros of the denominator yield vertical asymptotes. The following example demonstrates when this may not be the case.

Example 3.3: Determining Infinite Limits Analytically

Let

$$f(x) = \frac{x^2 - x}{x^2 - 4x + 3}$$

and evaluate the following limits analytically and find all vertical asymptotes.

a.) $\lim_{x \to 1} f(x)$

Solution. First re-write $f(x)$ in factored form

$$f(x) = \frac{x^2 - x}{x^2 - 4x + 3} = \frac{x(x-1)}{(x-1)(x-3)}$$

We see that the term $(x - 1)$ cancels from both the numerator and denominator giving us

$$f(x) = \frac{x\cancel{(x-1)}}{\cancel{(x-1)}(x-3)} = \frac{x}{x-3}$$

From algebra, we know that the point $x = 1$ is a "hole" in the graph of $f(x)$. Examining the limit here we see that

$$\lim_{x \to 1} \frac{x}{x-3} = \frac{1}{1-3} = -\frac{1}{2}$$

Since this value is not $\pm\infty$ (i.e. is a finite value) the point $x = 1$ is *not* a vertical asymptote of $f(x)$.

b.) $\lim_{x \to 3^-} f(x)$

Solution. We are approaching $x = 3$ from the left. For $x < 3$ we have

$$x < 3 \implies x - 3 < 0$$

This means we have a negative denominator that is getting closer and closer to zero. Note that close to 3 our numerator will be positive. Thus, we have a positive number divided by a smaller and smaller negative number. This tells us that our value of $f(x)$ is approaching a larger and larger (in magnitude) negative number. Therefore,

$$\lim_{x \to 3^-} f(x) \to -\infty$$

c.) $\lim_{x \to 3^+} f(x)$

Solution. Now we are approaching $x = 3$ from the right. Now for $x > 3$ we have

$$x > 3 \implies x - 3 > 0$$

This means we have a positive denominator that is getting closer and closer to zero. Again, close to 3 our numerator will be positive. Thus, we have a positive number divided by a smaller and smaller positive number. This tells us that our value of $f(x)$ is approaching a larger and larger positive number. Therefore,

$$\lim_{x \to 3^+} f(x) \to \infty$$

Thus, since $\lim_{x \to 3^-} f(x) \to -\infty \neq \lim_{x \to 3^+} f(x) \to \infty$, it follows that $\lim_{x \to 3} f(x)$ does not exist but that $x = 3$ is a vertical asymptote.

Lecture 4

Continuity

Section
2.4

How Continuity is Defined

So far in your study of mathematics, continuity has been defined as being able to draw the graph of a function from start to finish without picking up your pencil. In other words, a function is continuous if its graph has no holes or breaks in it. We can now define continuity more precisely with our knowledge of limits. A function can be continuous at a single point, or on an interval.

Continuity at a Point

We will start with a discussion of continuity of f at a single point a.

Definition 4.1: Continuity at a Point

- **Continuity at a Point**: A function $f(x)$ is said to be **continuous at a point** given by $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

In other words we must meet *all* of the following conditions:

1. $f(a)$ exists;
2. $\lim_{x \to a} f(x)$ exists;
3. $\lim_{x \to a} f(x) = f(a)$;

If even one of these cases fails, then $f(x)$ fails to be continuous at a and $x = a$ is known as a **point of discontinuity**.

This definition of continuity requires that *both* $\lim_{x \to a} f(x)$ and $f(a)$ exist. If one fails to exist then $f(x)$ cannot be continuous there. So to establish if a function $f(x)$ is continuous at a point a you need to verify all three statements. This leads us to the following theorem.

Theorem 4.1

If $f(x)$ is continuous at $x = a$ then

$$\lim_{x \to a} f(x) = f(a), \qquad \lim_{x \to a^-} f(x) = f(a), \qquad \lim_{x \to a^+} f(x) = f(a)$$

Note that this theorem is telling us that if we know f is continuous at a then we know the value of the three limits that are listed.

Example 4.1: Determining Continuity from a Graph

Examine the graph in Figure 4.1 and determine if $f(x)$ is continuous at the values

$$x = 1, \quad x = 2, \quad \text{and} \quad x = 4$$

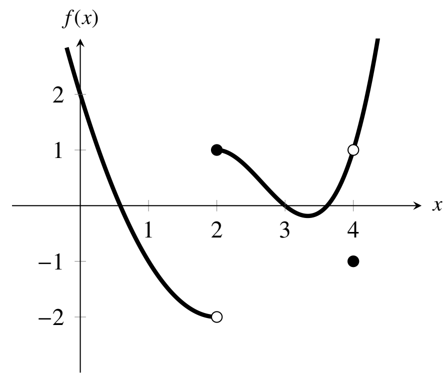


Fig. 4.1: Graph of $f(x)$

a.) $x = 1$

Solution. We see that $f(1) = -1$ and that

$$\lim_{x\to 1^-} f(x) = -1 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = -1$$

And so we see that the value of the function at $x = 1$ and the limit as $x \to 1$ are the same. In other words, we have

$$\lim_{x\to 1} f(x) = f(1)$$

Therefore, since f meets the 3 conditions of the definition then $f(x)$ is continuous at $x = 1$.

b.) $x = 2$

Solution. We see that $f(2) = 1$ but

$$\lim_{x\to 2^-} f(x) = -2 \quad \text{and} \quad \lim_{x\to 2^+} f(x) = 1$$

Since the left and right hand limits are not equal. In other words,

$$\lim_{x\to 2} f(x) \text{ does not exist.}$$

Since f fails the second condition of the definition, it is not continuous at $x = 2$.

This is known as a **jump discontinuity**. We can see from the graph that at $x = 2$ the graph "jumps" from -1 to 1.

c.) $x = 4$

Solution. We see that

$$f(4) = -1 \quad \text{and} \quad \lim_{x\to 4} f(x) = 1$$

We see that the function value is not equal to the value of the limit. So f fails the third part of the definition at $x = 4$. Therefore, $f(x)$ is not continuous at $x = 4$.

This type of discontinuity is known as a **removable discontinuity**. We can see from the graph that at $x = 4$ we have a "hole" in the graph.

In this example we got the limit values from the graph. Without a graph we would calculate the limit using strategies we have learned so far.

Continuity on an Interval

Definition 4.2: Continuity on an Interval

- **On an Open Interval:** A function $f(x)$ is said to be **continuous on an open interval** (a, b) if it is continuous at each point on the interval.
- **On an Closed Interval:** A function is **continuous on a closed interval** $[a, b]$ if it is continuous on the open interval (a, b) and if
  - The function f is **continuous from the right** at a, i.e. if

    $$\lim_{x\to a^+} f(x) = f(a)$$

    and
  - The function f is **continuous from the left** at b, i.e. if

    $$\lim_{x\to b^-} f(x) = f(b)$$

We will not spend any time specifically dealing with continuity on an interval. Since we have seen how to show a function f is continuous at a point a we can simply extend this to continuity on open and closed intervals.

Properties of Continuity

Now for a discussion of some of the properties of continuity. The results presented here are really just a summary of knowledge we already have from algebra.

Theorem 4.2: Continuity Rules

If f and g are continuous at a then the following functions are also continuous at a. Assume c is a constant and $n > 0$ is an integer.

$$f + g \qquad f - g \qquad cf$$

$$fg \qquad (f(x))^n \qquad \frac{f}{g}, \text{ provided } g(a) \neq 0$$

The next theorem is a consequence of these rules and summarizes what we already know about polynomial and rational functions in terms of their continuity.

Theorem 4.3: Continuity of Polynomial & Rational Functions

- A polynomial function is continuous for all x.
- A rational function of the form $\frac{p}{q}$ where p and q are polynomials, is continuous for all x for which $q(x) \neq 0$.

This next property is essential for evaluating limits of composite functions which we will see in the next section.

Theorem 4.4: Continuity of Composite Functions

If g is continuous at a and f is continuous at $g(a)$, then the composite function given by

$$(f \circ g)(x) = f(g(x))$$

is also continuous at a.

Example 4.2: Applying Continuity Rules

For what values of x is

$$f(x) = \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

continuous?

Solution. This is a rational function. The values not in the domain of this function are those for which the denominator is equal to zero. Setting the denominator equal to zero we see that

$$5 - 3x = 0 \quad \Longrightarrow \quad x = \frac{5}{3}$$

and so we must have that $x \neq \frac{5}{3}$.
Thus, by Theorem 4.3 we know that $f(x)$ is continuous everywhere except the point $x = \frac{5}{3}$.

Additionally, we can discuss the continuity of transcendental functions on their domains. The following theorem states the continuity of these functions.

Theorem 4.5: Continuity of Transcendental Functions

The following functions are continuous on their domains:

- **Trigonometric Functions:** $\sin x$, $\cos x$, $\tan x$, $\csc x$, $\sec x$, $\cot x$.
- **Inverse Trig Functions:** $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, etc.
- **Exponential Functions:** b^x, e^x
- **Logarithmic Functions:** $\log_b(x)$, $\ln(x)$

Limit of a Function Composition

Now that we have established a more precise definition for a function to be continuous and discussed some properties of continuous functions we can discuss the limit of a function composition.

Theorem 4.6: Limit of Function Composition

If $f(x)$ is continuous at $x = b$ and $\lim\limits_{x \to a} g(x) = b$ then,

$$\lim_{x \to a} f\left(g(x)\right) = f\left(\lim_{x \to a} g(x)\right)$$

Example 4.3: Limit of a Function Composition

Find $\lim_{x \to 0} e^{(\sin x)}$.

Solution. We know that $e^{(\sin x)}$ is the composition of the functions e^x and $\sin x$. By the previous theorem we have

$$\lim_{x \to 0} e^{(\sin x)} = e^{\left(\lim_{x \to 0} \sin x\right)}$$

We know that $\lim_{x \to 0} \sin x = \sin(0) = 0$. Thus by the previous theorem we have

$$\lim_{x \to 0} e^{(\sin x)} = e^{\left(\lim_{x \to 0} \sin x\right)} = e^0 = 1$$

The Intermediate Value Theorem

A common problem addressed in mathematics involves finding the roots of a function. Recall that the *roots* of a function f are also referred to as *zeros* of a function f or as *solutions* to $f(x) = 0$. These are all equivalent statements.

Suppose we have a continuous function f for which we want to find the roots.

If we set $f(x) = 0$ then if $f(a)$ and $f(b)$ are of opposite signs (regardless of their value), we know that the function had to cross the x axis at some point on the interval.

This implies that at some point the function was equal to zero at least once on the interval, thus locating a solution to the equation. An example of this is pictured in Figure 4.2.

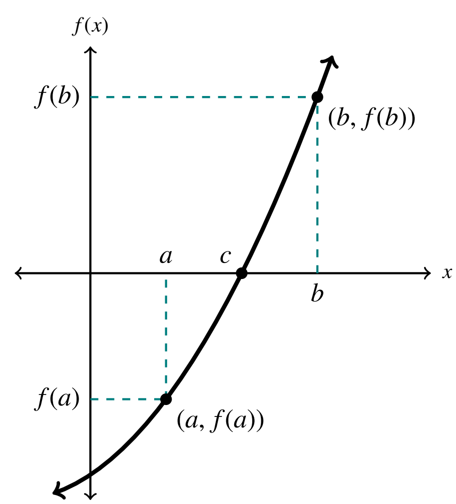


Fig. 4.2: Finding $f(c) = 0$ on (a, b)

While this may seem like a straightforward problem at this level, in the real world and in higher level mathematics, we don't know information about roots (or even how to find them) for many functions. So before going through a bunch of work to find the zeros of a function f it would be beneficial to know if they even exist!

In many situations it does not even matter what the actual value of the solution is, it is enough to know that at least one exists on an interval. Determining the location of a root (or solution to an equation) is made possible by the Intermediate Value Theorem.

Theorem 4.7: Intermediate Value Theorem

Suppose $f(x)$ is continuous on a closed interval $[a, b]$ and that $f(a) \neq f(b)$. Let L be any number between $f(a)$ and $f(b)$. Then there exists at least one number c such that

$$a < c < b \qquad \text{and} \qquad f(c) = L$$

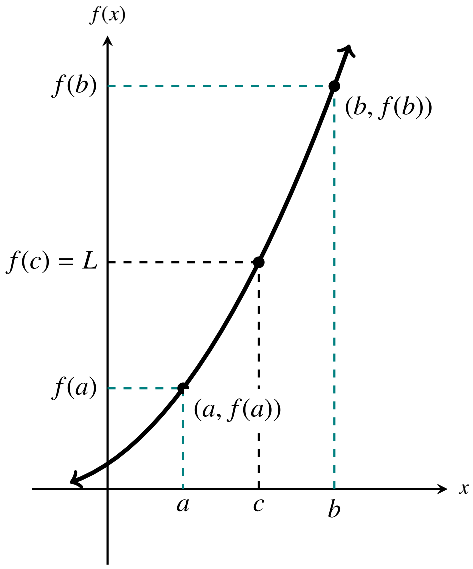


Fig. 4.3: Visual of the Intermediate Value Theorem

This theorem tells us that a *continuous* function on an interval $[a, b]$ will take on all values between $f(a)$ and $f(b)$.

Note that we talk about values between $f(a)$ and $f(b)$. This means we can either have the case that $f(a) < f(b)$ or we can have the case that $f(b) < f(a)$.

A visual representation of what the Intermediate Value Theorem states can be seen in Figure 4.3. Here we have that $f(a) < f(b)$ on the interval (a, b)

Hopefully, from this figure it is clear that if f is discontinuous at any point in the interval $[a, b]$ (i.e. there is a "break" in our graph), then we can not determine that such a value L exists. In other words, continuity of f is necessary for us to use the Intermediate Value Theorem.

You should also note that the Intermediate Value Theorem tells us that there is *at least* one value L. This means that it is possible for there to be *more* than one place where f has the value L.

We will see a formal example of the application of the Intermediate Value Theorem in the following example.

Example 4.4: Applying the Intermediate Value Theorem

Show

$$p(x) = 2x^3 - 5x^2 - 10x + 5$$

has a root in $[-1, 2]$.

Solution. First, we know that p is continuous since it is a polynomial and polynomials are continuous everywhere. Next we want to know if $p(x) = 0$ between $x = -1$ and $x = 2$. In other words, we want a number c for which

$$-1 < c < 2 \qquad \text{and} \qquad p(c) = 0$$

so we want $p(-1) < 0 < p(2)$ or equivalently $p(2) < 0 < p(-1)$. We see that $p(-1) = 8$ and $p(2) = -19$. So we have

$$-19 = p(2) < 0 < p(-1) = 8$$

Thus, $p(x) = 0$ somewhere between $p(-1)$ and $p(2)$. So by the Intermediate Value Theorem, there must be a number c in this interval for which $p(c) = 0$. Therefore, a root exists in the interval $[-1, 2]$.

Remark: With our knowledge of algebra, we know that we could attempt to factor this polynomial to find its roots. However, not all polynomials are able to be factored. Furthermore, the equation for finding solutions to cubics is far more complicated than our friend the quadratic formula. Plus, there doesn't even exist formulas for polynomials with degree higher than 4! This is where the Intermediate Value Theorem comes in handy.

Continuity is an important property of functions. Many operations or processes cannot be carried out on a function unless it is continuous. It is a key component in establishing the existence of a *derivative*, the main topic studied in the first semester of calculus.

Lecture 5

Definition of the Derivative

Sections
3.1.1-3.1.4; 3.2.1

Tangent Lines

There are two classic problems in calculus. The tangent line problem and the area problem. Now that we know some things about limits we are prepared to talk about the tangent line problem.

Definition 5.1: Tangent Line

Graphically, a **tangent line** can be described as the line that touches, but does not cross a graph at a certain point. This is demonstrated in Figure 5.1.

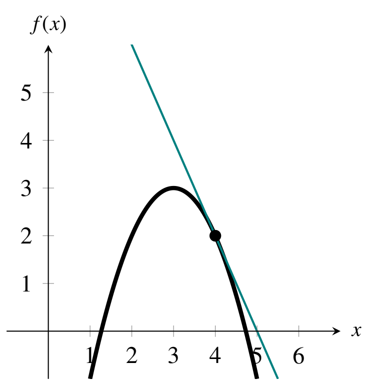


Fig. 5.1: A tangent line

Recall that to find the equation of a line we need the slope of the line and a point on the line. Given a slope m and a point on the line (x_1, y_1) we can construct the **point-slope** form of a line. Recall that the point-slope form of a line is given by

$$y - y_1 = m(x - x_1)$$

So in order to find a line tangent to a graph at a certain point we need to find the slope of the tangent line at that point.

In algebra, the slope and equation of a line is usually discussed using the points (x_1, y_1) and (x_2, y_2). Here we will instead use the points $(x, f(x))$ and $(x + h, f(x + h))$. In other words, the slope of a line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h}$$

We can approximate the tangent line using the something called the **secant line**. The definition of a secant line is given below.

Definition 5.2: Secant Line

A line drawn between any two points $(x, f(x))$ and $(x + h, f(x + h))$ on a graph is known as a **secant line**. The secant line is pictured in Figure 5.2.

The slope of the secant line is then given by the same slope formula,

$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{h}$$

When the slope formula is seen in this form it is sometimes referred to as the **difference quotient**. It is a crucial piece when discussing the definition of the derivative.

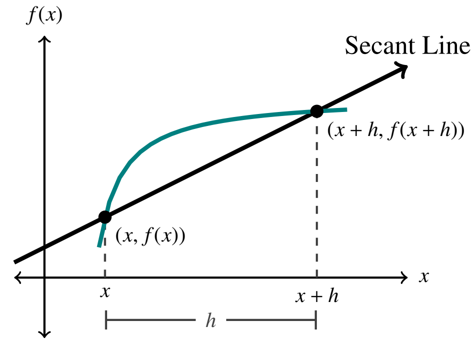


Fig. 5.2: Slope of the secant line

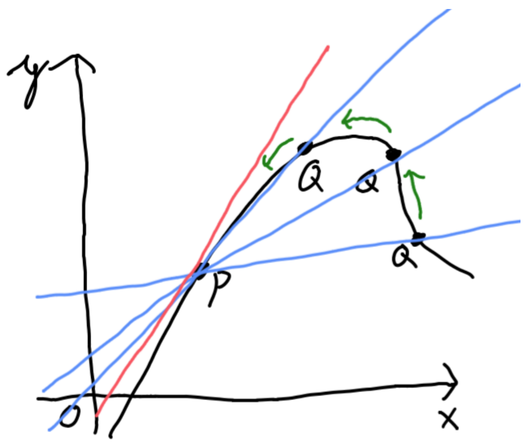


Fig. 5.3: Secant and Tangent lines

The secant line can help us approximate the tangent line. How can we do this? Take the graph of the function shown in Figure 5.3. A single secant line defined by the points P and Q does not do a good job of estimating the tangent line given at point P.

What if we move Q closer to the point P? The next secant line will have a better approximation to the tangent line. If we keep doing this with points that are closer and closer together, eventually the distance between them will be zero. This will give us the slope of the tangent line.

What we really want is for the distance h in Fig. 5.2 to be as small as possible. The smallest h could be is of course zero. We know that we can't divide by zero, so what do we do? This is where we can use our newly found knowledge of limits! By evaluating the limit as h goes to zero we will be able to find the slope of our tangent line.

Definition 5.3: Slope of the Tangent Line

For any point x in the domain of a function f, the **slope of the tangent line**, denoted $m_{\tan}$, is given by

$$m_{\tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

What if we want to know the value of the slope of the tangent line for a function given any value x? It would be nice to find an equation for which we could just plug in a value and obtain our slope. The equation that will give us this information is known as the **derivative**.

Definition of the Derivative

Instead of finding the slope of the line tangent to f at a single point every time we would like to have a *function* that will allow us to find this information more easily. Luckily, we can find the function governing the slope of the tangent line using the same idea given above to calculate the slope of the tangent line.

This means we can find an equation that will give us the value of the slope along the entire graph of the function f. This is known as the **derivative of a function** f.

Definition 5.4: Derivative of a Function

The **derivative** of f is the function given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

provided that the limit exists and that x is in the domain of f.

The symbol f' above is read as "f prime". This is only one way to denote the derivative. There are actually several different ways to denote the derivative of a function which we will discuss later. First, we will look at a simple example of how to apply the definition of the derivative.

Example 5.1: Compute the Derivative using the Definition

Find the derivative of the function $g(x) = 2x + 3$.

Solution. We need to use the definition of the derivative of a function given in Definition 5.4. In other words we want to find

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

For more complicated steps it can sometimes be helpful to calculate $g(x+h)$ before proceeding with your limit evaluation. So we note that

$$g(x+h) = 2(x+h) + 3 = 2x + 2h + 3$$

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{[2x+2h+3] - [2x+3]}{h} && \text{Plug in } g(x+h) \text{ \& } g(x) \\
&= \lim_{h\to 0} \frac{2x+2h+3-2x-3}{h} && \text{Distribute } -1 \text{ to terms of } g(x) \\
&= \lim_{h\to 0} \frac{\cancel{2x}+2h+\cancel{3}-\cancel{2x}-\cancel{3}}{h} && \text{Combine like terms} \\
&= \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}} && \text{Cancel common terms} \\
&= \lim_{h\to 0} 2 = 2
\end{aligned}$$

Therefore, $g'(x) = 2$.

Now lets look at some slightly more complicated examples of computing the derivative.

Example 5.2: Computing Derivatives using the Definition

Compute the derivative of the following functions using the definition.

a.) $f(x) = x^2 - x$

Solution. We need to use the definition of the derivative of a function given in Definition 5.4. So we want to find

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

For complicated functions it can sometimes be helpful to calculate $f(x+h)$ before proceeding with your limit evaluation. So we note that

$$f(x+h) = (x+h)^2 - (x+h) = x^2 + 2hx + h^2 - x - h$$

Now we find $f'(x)$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{[x^2 + 2hx + h^2 - x - h] - [x^2 - x]}{h} && \text{Plug in } f(x+h) \text{ \& } f(x) \\
&= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x - h - x^2 + x}{h} && \text{Distribute } -1 \text{ to terms of } f(x) \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x} - h - \cancel{x^2} + \cancel{x}}{h} && \text{Combine \& cancel common terms} \\
&= \lim_{h\to 0} \frac{2hx + h^2 - h}{h} && \text{Factor out } h \\
&= \lim_{h\to 0} \frac{\cancel{h}(h + 2x - 1)}{\cancel{h}} && \text{Cancel common terms} \\
&= \lim_{h\to 0} (h + 2x - 1) \\
&= (0) + 2x - 1 && \text{Direct substitution} \\
&= 2x - 1
\end{aligned}$$

b.) $f(x) = \sqrt{9 - x}$

Solution. We again use the definition of the derivative given in Definition 5.4. It's important not to forget strategies we have learned for evaluating limits. Here we will need to multiply by the conjugate.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{9 - (x+h)} - \sqrt{9 - x}}{h} \\
&= \lim_{h\to 0} \left(\frac{\sqrt{9 - (x+h)} - \sqrt{9 - x}}{h} \right) \left(\frac{\sqrt{9 - (x+h)} + \sqrt{9 - x}}{\sqrt{9 - (x+h)} + \sqrt{9 - x}} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{(9-(x+h))-(9-x)}{h(\sqrt{9-(x+h)}+\sqrt{9-x})} \\
&= \lim_{h\to 0} \frac{\cancel{9}-\cancel{x}-h-\cancel{9}+\cancel{x}}{h(\sqrt{9-(x+h)}+\sqrt{9-x})} \\
&= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(\sqrt{9-(x+h)}+\sqrt{9-x})} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{9-(x+h)}+\sqrt{9-x}} \\
&= \frac{-1}{\sqrt{9-(x+(0))}+\sqrt{9-x}} \\
&= \frac{-1}{\sqrt{9-x}+\sqrt{9-x}} \\
&= \frac{-1}{2\sqrt{9-x}}
\end{aligned}
$$

Derivative Notation

Indicating the derivative of a function f by $f'(x)$ is known as "prime" notation. The textbook mostly uses $\frac{d}{dx}$ notation to indicate the derivative, this is known as Leibniz notation. Several different notations exist for the derivative because Calculus was independently developed by several different people at the same time in history.

Each type of notation can be more useful in some situations over others. Often, it is just the person's preference as to which is used. I tend to stick to prime notation in most cases. I find it more simple to use and it leaves less letters to keep track of! Ultimately, it's up to you as to which you use, but you should be able to recognize all of them.

Here is a summary of common derivative notations used.

- $f'(x)$ is read as "f prime of x"
- $\frac{dy}{dx}$ is read as "derivative of y with respect to x"
- y' is read as "y prime"
- $\frac{d}{dx}[f(x)]$ is read as "derivative of $f(x)$ with respect to x"

Finding Tangent Lines

Since we can obtain $f'(x)$, where x represents some real number, we can of course find $f'(a)$ for a specific value a. This leads us to an alternative form of the definition of the derivative.

Definition 5.5: Definition of the Derivative at a Point

The **derivative** of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Why have another form? This form gives us a convenient way to find the derivative using the definition at a single point. It can be much easier to use in certain situations like showing differentiability of a function discussed in the next lecture.

At this point in the course, the main take away is the following statement.

Theorem 5.1: Derivative is the Slope of the Tangent Line

The slope of the tangent line to the graph of the function $y = f(x)$ at the point $(a, f(a))$ is $f'(a)$.

We know that the slope of a tangent line at a point $(a, f(a))$ is given by $f'(a)$. The general equation of a tangent line for any function $f(x)$ at $x = a$ can be defined as follows.

Definition 5.6: Equation of a Tangent Line at a Point

The **equation of the tangent line** at the point $(a, f(a))$ is given by

$$y = f(a) + f'(a)(x - a)$$

We now examine an example of how to calculate the tangent line for a given function at a specified point.

Example 5.3: Finding the Equation of a Tangent Line

Find an equation of the line tangent to the function $f(x) = 9 - 2x^2$ at the point $(2, 1)$.

Solution. We want to find the slope of the tangent line when $x = 2$, i.e. using Definition 5.5 we have $a = 2$. Now we evaluate this limit.

$$\begin{aligned}
\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} &= \lim_{x \to 2} \frac{(9 - 2x^2) - (1)}{x - 2} \\
&= \lim_{x \to 2} \frac{8 - 2x^2}{x - 2} \\
&= \lim_{x \to 2} \frac{-2\cancel{(x-2)}(x + 2)}{\cancel{x - 2}} \\
&= \lim_{x \to 2} [-2(x + 2)] = -2 \cdot 4 = -8
\end{aligned}$$

So we have found that $f'(2) = -8$, this is the slope of our tangent line at the point $(2, 1)$. To find the equation of this line we simply plug it into the equation given in Definition 5.6 and obtain

$$y = 1 - 8(x - 2) \implies y = -8x + 17$$

this is the equation of the line tangent to $f(x)$ at the point $(2, 1)$.

Lecture 6

Differentiability

Sections
3.2.1, 3.2.3-3.2.4

Differentiability

When a function has a derivative on an interval or at a point it is said to be *differentiable* there.

Definition 6.1: Differentiability

A function f is

- **differentiable at a point** x if its derivative exists at x.
- **differentiable on an interval** (a,b) if it is differentiable at every point in the interval.

Now that we know how to find a derivative using the definition we can talk about when we are able to actually do this. Since the definition involves a limit, we will only be able to compute derivatives when this limit exists. Previously, we have seen some cases when the limit does not exist. These same cases will affect our ability to find a derivative at a point or along an interval.

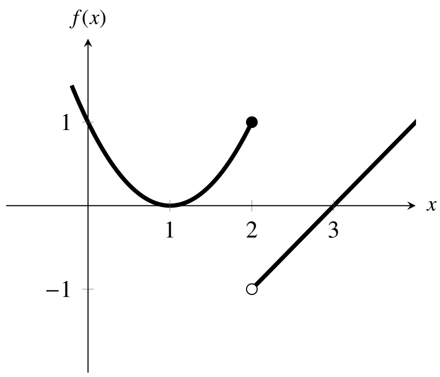
When is a function not differentiable?

Just as before, in order for this limit to exist we need both the left hand limit and the right hand limit to be equal. We will use the alternative form of the derivative in the following examples. In other words, for f to be differentiable at a point $x = c$ we will need

$$\lim_{x\to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c^+} \frac{f(x) - f(c)}{x - c}$$

We will refer to these expressions as the **derivative from the left** and the **derivative from the right** respectively.

Example 6.1: Discontinuous Functions



Consider the function

$$f(x) = \begin{cases} (x-1)^2, & \text{if } x \le 2 \\ x-3, & \text{if } x > 2 \end{cases}$$

Note that

$$f(2) = (2-1)^2 = 1$$

Is f differentiable at $x = 2$?

Solution. To determine differentiability we will examine the derivative from the left and the derivative from the right at $a = 2$.

Derivative from the left:

$$\begin{aligned} \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} &= \lim_{x\to 2^-} \frac{(x-1)^2-1}{x-2} \\ &= \lim_{x\to 2^-} \frac{x^2-2x}{x-2} && \text{Expand and combine like terms} \\ &= \lim_{x\to 2^-} \frac{x(x-2)}{x-2} && \text{Factor out common terms} \\ &= \lim_{x\to 2^-} x && \text{Cancel common terms} \\ &= 2 && \text{Direct Substitution} \end{aligned}$$

Derivative from the right:

$$\begin{aligned} \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} &= \lim_{x\to 2^+} \frac{(x-3)-1}{x-2} \\ &= \lim_{x\to 2^+} \frac{x-4}{x-2} && \text{Expand and combine like terms} \\ &= \frac{-2}{+\text{small number}} \\ &= -\infty \end{aligned}$$

Since

$$\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = 2 \ne \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = -\infty$$

then $\lim_{x\to 2} \frac{f(x)-f(2)}{x-2}$ does not exist. Therefore, $f(x)$ is *not* differentiable at $x = 2$.

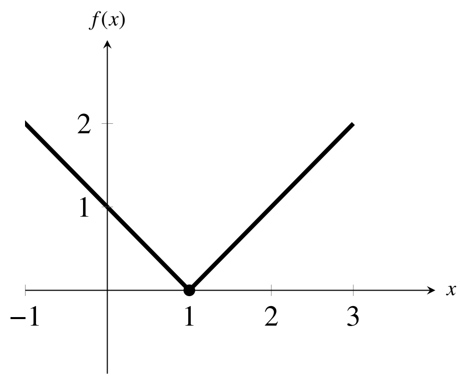
We saw in the previous example that a discontinuous function was not differentiable. This turns out to be the case for any discontinuous function. This gives us the following theorem.

Theorem 6.1: Not Continuous Implies Not Differentiable

If f is not continuous at a, then f is not differentiable at a.

When we have functions with sharp turns or "cusps" if we tried to draw a tangent line at that point there would be an infinite amount of possibilities for our slope. There isn't a single unique line that would be tangent to the curve at the point. The next example demonstrates this case.

Example 6.2: Function with a Sharp Point



Consider the function $g(x) = |x - 1|$.

Solution. We can see that the function is continuous at $x = 1$ but

$$\lim_{x \to 1^+} \frac{g(x) - g(1)}{x - 1} = \lim_{x \to 1^+} \frac{|x-1| - (0)}{x - 1} = \lim_{x \to 1^+} \frac{(x-1)}{x-1} = 1$$

and

$$\lim_{x \to 1^-} \frac{g(x) - g(1)}{x - 1} = \lim_{x \to 1^-} \frac{|x-1| - (0)}{x - 1} = \lim_{x \to 1^-} \frac{-(x-1)}{x-1} = -1$$

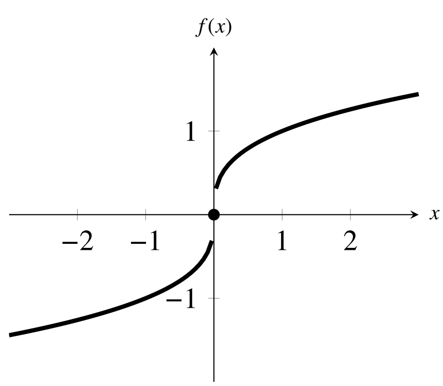
We see that

$$\lim_{x \to 1^-} \frac{g(x) - g(1)}{x - 1} \neq \lim_{x \to 1^+} \frac{g(x) - g(1)}{x - 1}$$

so $\lim\limits_{x \to 1} \dfrac{g(x) - g(1)}{x - 1}$ does not exist. Therefore, $g(x)$ is *not differentiable* at $x = 1$.

Next we consider a graph that has a vertical tangent line at a certain point.

Example 6.3: Function with a Vertical Tangent Line



Consider the function $h(x) = x^{1/3}$.

Solution. This function is continuous at $x = 0$. However,

$$\lim_{x \to 0} \frac{h(x) - h(0)}{x - 0} = \lim_{x \to 0} \frac{x^{1/3} - (0)}{x} = \lim_{x \to 0} \frac{1}{x^{2/3}} = \infty$$

This means that $\lim\limits_{x \to 0} \dfrac{h(x) - h(0)}{x - 1}$ does not exist. Therefore, $h(x)$ is *not differentiable* at $x = 0$.

What can we say about differentiable functions? It turns out that differentiability is closely linked to whether or not a function is continuous, either at a point or on an interval.

Theorem 6.2: Differentiable Implies Continuous

If f is differentiable at a, then f is continuous at a.

In other words, if a function is differentiable at a point or on an interval, then it must be continuous at that point or on that interval.

In Examples 6.2 and 6.3 we saw examples of continuous functions that failed to be differentiable at a point. This means that converse of Theorem 6.2 is *not* true. In other words, a continuous function can fail to be differentiable. This means that just because a function is continuous, it does not necessarily mean it is differentiable there.

We have now explored where the derivative comes from. Using the definition to find derivatives can be a tedious process. In the next lectures we will learn about some important rules that provide us with a more efficient way to find derivatives.

Lecture 7

Basic Differentiation Rules

Sections
3.3.1-3.3.2, 3.3.5

Introduction

The limit definition is cumbersome to use every time you need to calculate a derivative. The next few lectures discuss formulas that allow us to find the derivative directly without bothering with limits.

It is important to note that each of these formulas comes directly from the definition of the derivative. This is why it was so important to learn before. The differentiation rules also tend to tell us nothing about differentiability. Just because we can find a formula using the differentiation rules does not mean that our derivative will exist everywhere!

We will demonstrate some of the proofs for the differentiation rules. If interested, the proofs of the remaining results can easily be found either in your textbook or by doing a quick internet search.

The Constant Rule

The derivative of a constant is the easiest of the differentiation rules to show.

Theorem 7.1: Derivative of a Constant

Let c be a real number, then

$$\frac{d}{dx}[c] = 0$$

Proof. Let $f(x) = c$. Then, by the limit definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} \frac{0}{h} = 0 \end{aligned}$$

□

This result can be seen in the graph of a constant function. Recall that the slope of a constant function is 0 so any tangent line must also have a slope of zero.

The Power Rule

The next rule is by far the most common derivative rule you will use.

Theorem 7.2: The Power Rule

Let n be a real number then

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

for all x where x^n and x^{n-1} are both defined.

This rule works for $n < 0$ (i.e. when n is negative), $n = 1$, n is rational (i.e. $n = \frac{a}{b}$ where $a, b \in \mathbb{R}$ and $b \neq 0$), or n is irrational (i.e. a number line π). The textbook presents this rule separately for each case. For simplicity, we present it once here and will also skip the proof of this rule for the same reason.

The main thing to take away at this point is to see the pattern that this rule follows. Essentially, we are multiplying our term by the value of the exponent and then subtracting 1 from the exponent. For example, when we use the power rule on the function $f(x) = x$ we have

$$f'(x) = (1)x^{1-1} = x^0 = 1$$

In fact, the constant rule is just a special case of the power rule. If $f(x) = c$, then $f(x) = cx^0$, so

$$f'(x) = 0cx^{0-1} = 0cx^{-1} = \frac{0c}{x} = \frac{0}{x} = 0$$

Example 7.1: Applying the Power Rule

Using the power rule. Find the derivative of the following functions

a.) $f(x) = x^3$

Solution.

$$f'(x) = \frac{d}{dx}[x^3] = 3x^{3-1} = 3x^2$$

b.) $g(x) = \dfrac{1}{x^4}$

Solution. In order to use the Power Rule we want the function to have the form $g(x) = x^n$ so we rewrite g

$$g(x) = \frac{1}{x^4} = x^{-4}$$

Now we can apply the Power Rule to find the derivative

$$g'(x) = \frac{d}{dx}[x^{-4}] = -4x^{-4-1} = -4x^{-5} = -4\left(\frac{1}{x^5}\right) = \frac{-4}{x^5}$$

c.) $h(x) = \sqrt[3]{x^4}$

Solution. Rewriting as a single exponent we have

$$h(x) = \sqrt[3]{x^4} = x^{\frac{4}{3}}$$

Taking the derivative

$$h'(x) = \frac{d}{dx}[x^{\frac{4}{3}}] = \frac{4}{3}x^{\frac{4}{3}-1} = \frac{4}{3}x^{\frac{4}{3}-\frac{3}{3}} = \frac{4}{3}x^{\frac{1}{3}} = \frac{4}{3}\sqrt[3]{x}$$

The Constant Multiple Rule

Theorem 7.3: Constant Multiple Rule

If f is differentiable function and c is a real number, then

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

Proof.

$$\begin{aligned}
\frac{d}{dx}[cf(x)] &= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} && \text{Definition of Derivative}\\
&= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right]\\
&= c\left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right] && \text{Apply Theorem 1.3}\\
&= cf'(x)
\end{aligned}$$

□

Example 7.2: Applying the Constant Multiple Rule

Differentiate the following functions

a.) $f(x) = \dfrac{7}{x}$

Solution. We first rewrite $f(x)$ so that it has an exponent.

$$f(x) = \frac{7}{x} = 7\left(\frac{1}{x}\right) = 7x^{-1}$$

Now differentiating we obtain

$$\begin{aligned} f'(x) &= \frac{d}{dx}[7x^{-1}] = 7\frac{d}{dx}[x^{-1}] && \text{Applying Constant Multiple Rule} \\ &= 7(-1)x^{-1-1} = -7x^{-2} && \text{Applying Power Rule} \\ &= \frac{-7}{x^2} \end{aligned}$$

b.) $g(x) = \dfrac{x^3}{2}$

Solution. Differentiating, we have

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left[\frac{x^3}{2}\right] = \frac{1}{2}\frac{d}{dx}\left[x^3\right] && \text{Applying Constant Multiple Rule} \\ &= \frac{1}{2}(3)x^{3-1} = \frac{3}{2}x^2 && \text{Applying Power Rule} \end{aligned}$$

Sum and Difference Rules

Theorem 7.4: The Sum and Difference Rules

Let f and g be differentiable functions then

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

We show the proof the sum rule. The difference rule is proved similarly.

Proof.

$$\begin{aligned} \frac{d}{dx}[f(x) + g(x)] &= \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\ &= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right] \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x) \end{aligned}$$

□

Example 7.3: Applying Sum and Difference Rules

Differentiate the function $f(x) = \frac{-x^4}{2} + 3x^3 - 2x + 7$

Solution.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\frac{-x^4}{2} + 3x^3 - 2x + 7\right] \\ &= \frac{d}{dx}\left[\frac{-x^4}{2}\right] + \frac{d}{dx}[3x^3] - \frac{d}{dx}[2x] + \frac{d}{dx}[7] && \text{Sum \& Difference Rule} \\ &= \frac{-1}{2}(4)x^{4-1} + 3(3)x^{3-1} - 2x^{1-1} + 0 && \text{Power \& Constant Rules} \\ &= \frac{-4}{2}x^3 + 9x^2 - 2 \\ &= -2x^3 + 9x^2 - 2 \end{aligned}$$

Finding Tangent Lines

At this point the most common application you will see is finding the equation for a tangent line. We have already seen how to do this using the definition. We now demonstrate this using our newly learned differentiation rules.

Example 7.4: Find the Equation of Line Tangent to the Graph

Find the line tangent to the curve $f(x) = -3x^2 + 2$ at $x = 1$.

Solution. Recall that the derivative is the slope of the tangent line and that the equation of the tangent line at a point $x = a$ given in Definition 5.6 is

$$y = f(a) + f'(a)(x - a)$$

Now that we have all of these differentiation rules, this derivative is simple to find. We have

$$f'(x) = -6x$$

This is the equation that gives the slope of the line at a given x value. Here we have $x = 1$ so

$$f'(1) = -6(1) = -6$$

We also need to find $f(1)$. We have

$$f(1) = -3(1)^2 + 2 = -3 + 2 = -1$$

We now have $f(1)$ and $f'(1)$ and so the equation of the tangent line is

$$y = -1 - 6(x - 1) \implies y = -6x + 5$$

Lecture 8

Advanced Differentiation Rules

Sections
3.2.5; 3.3.3-3.3.4, 3.3.6

Introduction

Now that we have seen some of the basic differentiation rules we can move on to finding derivatives of functions that are the product or quotient of 2 (or more) functions.

The Product Rule

The product of two functions f and g looks like $f \cdot g$ or $f(x) \cdot g(x)$. The following rule gives us a means to differentiate functions of this form.

Theorem 8.1: The Product Rule

Let f and g be differentiable functions then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Some mathematical proofs, such as the proof of the Sum Rule, are straightforward. Others involve "tricky tricks" that may appear unmotivated to a reader. This proof involves such a step where we subtract and add the same quantity, essentially adding nothing, to manipulate our limit. The added terms are show below in color.

Proof.

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h} \right] \\
&= \lim_{h\to 0} \left[f(x+h)\frac{g(x+h) - g(x)}{h} \right] + \lim_{h\to 0} \left[g(x)\frac{f(x+h) - f(x)}{h} \right] \\
&= \lim_{h\to 0} f(x+h) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

□

Remark: Your book does these derivatives in a process I call "brute force". It involves carrying out the derivatives at each step inline as you are doing your work. This process not only looks messy, but it is very difficult to catch your errors on intermediary derivative calculations. I organize my work differently than the book does when calculating derivatives. I have done my best to demonstrate this process in the following examples.

When I do derivatives that involve these more complicated rules I always write out the general version of the rule first. This not only helps keep things organized but also helps you to remember the rule! Derivatives which use this rule (and others) can get complicated very quickly. Often, they involve applying multiple rules. Having some type of organizational structure to your work not only ensures that I know what you are doing, but also makes it very easy to spot and correct mistakes you have made along the way! I will do my best here to demonstrate my method of writing up my derivative work.

Example 8.1: Applying the Product Rule

Differentiate the function $y = (2x + 7)(4x^2 + 3x)$

Solution. Although we could expand this and use the power rule we can get the derivative faster by using the product rule.

Since $\quad y' = \dfrac{d}{dx}[fg] = f'g + fg'$

$$\begin{array}{lll} \text{Let} & f = 2x + 7 \quad , & g = 4x^2 + 3x \\ & f' = 2 \quad , & g' = 8x + 3 \end{array}$$

So our derivative is

$$\begin{aligned} y' &= 2(4x^2 + 3x) + (2x + 7)(8x + 3) \\ &= 24x^2 + 68x + 21 \end{aligned}$$

Take care when simplifying and combining terms! Often, expanding your final answer is messy and does not help in your understanding. If you miss a step in the simplification, any subsequent questions answered using the derivative will be incorrect. At a minimum, you should always include the initial application of the rule so that it is easy to go back and check for errors.

The Quotient Rule

Recall that the quotient of two functions f and g has the general form

$$\frac{f}{g} = \frac{f(x)}{g(x)}$$

The next rule demonstrates how to differentiate functions of this form.

Theorem 8.2: The Quotient Rule

Let f and g be differentiable functions then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

Example 8.2: Using the Quotient Rule differentiate

Differentiate

$$y = \frac{5x-2}{x^2+1}$$

Solution.

Here we can apply the quotient rule.

Since $\quad y' = \dfrac{d}{dx}\left[\dfrac{f}{g}\right] = \dfrac{f'g - fg'}{g^2}$

Let $\quad f = 5x - 2 \quad , \quad g = x^2 + 1$
$\quad\quad f' = 5 \quad , \quad g' = 2x$

So our derivative is

$$y' = \frac{5(x^2+1) - (5x-2)(2x)}{(x^2+1)^2}$$

Higher Order Derivatives

So far we have learned an extensive toolbox for finding derivatives of most functions. Better yet, we see that we can avoid the use of the limit definition when calculating most derivatives. It is also possible to find higher order derivatives. A higher order derivative can be thought of as a "derivative of a derivative".

In order to find the nth order derivative you take the derivative of the $n-1$ derivative.

$$f^{(n)}(x) = \frac{d}{dx}[f^{(n-1)}(x)]$$

For example, to find the second derivative, $f''(x)$, we must take the derivative of the first derivative, $f'(x)$, i.e.

$$f''(x) = \frac{d}{dx}[f'(x)]$$

There are a few ways to denote higher order derivatives as seen in the table below.

First Derivative:	y',	$f'(x)$,	$\frac{dy}{dx}$,	$\frac{d}{dx}[f(x)]$
Second Derivative:	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$,	$\frac{d^2}{dx^2}[f(x)]$
Third Derivative:	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$,	$\frac{d^3}{dx^3}[f(x)]$
Fourth Derivative:	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{d^4y}{dx^4}$,	$\frac{d^4}{dx^4}[f(x)]$
$\vdots$				
***n*-th Derivative:**	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{d^ny}{dx^n}$,	$\frac{d^n}{dx^n}[f(x)]$

Table 8.1: Common Derivative Notation

It is important to note that when parenthesis are used around an exponent that this denotes the "nth order derivative of f".

Example 8.3: Higher order derivatives

Find the indicated derivative of $y = x^2 e^{4x}$. *Note:* The derivative of e^{4x} is $4e^{4x}$, a fact that will be explored further in the next lecture. This can be derived using the definition of the derivative.

a.) Find y'

Solution. Here we will need to apply the Product Rule.

Since $\quad y' = \dfrac{d}{dx}[x^2 e^{4x}] = \dfrac{d}{dx}[fg] = f'g + fg'$

Let $\quad f = x^2 \quad , \quad g = e^{4x}$
$\qquad f' = 2x \quad , \quad g' = 4e^{4x}$

So our *first* derivative is

$$y' = 2xe^{4x} + 4x^2 e^{4x} = e^{4x}(2x + 4x^2)$$

b.) Find y''

Solution. We first note that $y'' = [y']' = \dfrac{d}{dx}[e^{4x}(2x+4x^2)]$. We need to apply the Product Rule.

Since $\quad y'' = \dfrac{d}{dx}[e^{4x}(2x + 4x^2)] = \dfrac{d}{dx}[pq] = p'q + pq'$

Let $\quad p = e^{4x} \quad , \quad q = 2x + 4x^2$
$\qquad p' = 4e^{4x} \quad , \quad q' = 2 + 8x$

So our *second* derivative is

$$\begin{aligned} y'' &= 4e^{4x}(2x + 4x^2) + e^{4x}(2 + 8x) \\ &= e^{4x}[4(2x + 4x^2) + (2 + 8x)] \\ &= e^{4x}[16x^2 + 16x + 2] \end{aligned}$$

c.) Find y'''

Solution. We first note that $y''' = [y'']' = \dfrac{d}{dx}[e^{4x}(16x^2 + 16x + 2)]$. We see that we need to apply the Product Rule.

Since $\quad y''' = \dfrac{d}{dx}[e^{4x}(16x^2 + 16x + 2)] = \dfrac{d}{dx}[uv] = u'v + uv'$

Let $\quad u = e^{4x} \quad , \quad v = 16x^2 + 16x + 2$
$\qquad u' = 4e^{4x} \quad , \quad v' = 32x + 16$

So our *third* derivative is

$$y''' = 4e^{4x}[16x^2 + 16x + 2] + e^{4x}(32x + 16)$$

Remark: Note that we did not multiply everything out after finding y' and y''. Factoring out a common term at each step made our next derivative easier to find. Sometimes you may even find that it is not helpful to simplify before finding the next derivative. The main point is to not turn into a simplification robot! Think before you distribute terms or perform other simplifications. Not sure if a certain form will help you or not? Try it out on scratch paper! Don't be afraid to test things. A more complicated series of steps will still get the same answer but it takes longer and you may make mistakes.

Lecture 9

The Chain Rule and Derivatives of Transcendental Functions

Sections
3.5; 3.6; 3.9.1-3.9.2

The Chain Rule

Function compositions are commonly seen in the study of mathematics. Recall that a function composition is denoted by

$$(f \circ g)(x) = f(g(x))$$

For example the function

$$\sqrt{x^3 + 2x + 5}$$

Is the composition of the functions $f(x) = \sqrt{x}$ and $g(x) = x^3 + 2x + 5$. We have a special rule to find derivatives of these types of functions. This is known as the chain rule.

Theorem 9.1: The Chain Rule

Let $y = f(g(x))$ and $g(x)$ be differentiable functions. Then

$$y' = \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

In other words we take the derivative of the outside function multiplied by the derivative of the inside function.

Example 9.1: Using the Chain Rule

Differentiate the function $y = (x^2 + 1)^4$

Solution. We apply the Chain Rule by recognizing that our "outside" function as $f(x) = x^4$ and our "inside" function as $g(x) = x^2 + 1$

$$\text{Since} \quad y' = \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

$$\text{Let} \quad f(g) = g^4 \quad , \quad g(x) = x^2 + 1$$

$$f'(g) = 4g^3 \quad , \quad g'(x) = 2x$$

So our derivative is

$$\begin{aligned} y' &= 4(g(x))^3 \cdot g'(x) \\ &= 4(x^2 + 1)^3 \cdot (2x) \\ &= 8x(x^2 + 1)^3 \end{aligned}$$

Of course we could have also multiplied everything out and then taken the derivative of each term but this is more tedious than applying the Chain Rule.

The textbook defines a separate rule for applying the power rule with functions. This is really just an application of the chain rule. We find the derivative of the "outside" function using any of the usual rules we have for differentiation. There really is no need for a separate rule to be defined. The rule is defined below in case you have trouble seeing functions of this form.

Theorem 9.2: The General Power Rule

Let $y = [f(x)]^n$ where f is a differentiable function and n is a real number. Then

$$y' = n[f(x)]^{n-1} \cdot f'(x)$$

We are now able to take the derivatives of more complicated functions very quickly simply by rewriting them and applying the chain rule.

Example 9.2: More Applications of the Chain Rule

Differentiate the following functions.

a.) $y = \sqrt[3]{(x^2+1)^4}$.

Solution. First we rewrite the function with an exponent.

$$y = \sqrt[3]{(x^2+1)^4} = (x^2+1)^{\frac{4}{3}}$$

Now we can use the chain rule. Here I will be using u for my inside function. The letter u is often used as a substitution variable.

Since $\quad y' = \dfrac{d}{dx}[f(u)] = f'(u(x)) \cdot u'(x)$

$$\text{Let} \quad f(u) = u^{\frac{4}{3}} \quad , \quad u(x) = x^2 + 1$$
$$f'(u) = \frac{4}{3}u^{\frac{1}{3}} \quad , \quad u'(x) = 2x$$

Note that in our definition of f we treat u as a variable to simplify our derivatives and to ensure we have the correct derivative of the "outside" function. So our derivative is

$$\begin{aligned} y' &= \frac{4}{3}(u(x))^{\frac{1}{3}} \cdot u'(x) \\ &= \frac{4}{3}(x^2+1)^{\frac{1}{3}}(2x) \\ &= \frac{8x}{3}(x^2+1)^{\frac{1}{3}} \text{ or } \frac{8x}{3}\sqrt[3]{x^2+1} \end{aligned}$$

b.) $y = \dfrac{-7}{(2t-3)^2}$

Solution. Re-write with an exponent.

$$y = \frac{-7}{(2t-3)^2} = -7(2t-3)^{-2}$$

Now we apply Chain Rule

Since $y' = \dfrac{d}{dt}[f(u)] = f'(u(t)) \cdot u'(t)$

Let $f(u) = -7u^{-2}$, $u(t) = 2t-3$

$f'(u) = 14u^{-3}$, $u'(t) = 2$

So our derivative is

$$\begin{aligned} y' &= 14(u(t))^{-3} \cdot u'(t) \\ &= 14(2t-3)^{-3}(2) \\ &= 28(2t-3)^{-3} \text{ or } \frac{28}{(2t-3)^3} \end{aligned}$$

Remark: Since the denominator of any rational function can be written with a negative exponent it is possible to write any rational function as a product of functions. In some cases this results in a simpler calculation using the product rule (instead of the quotient rule). It is very beneficial to improve your ability to rewrite functions in new ways as it often drastically simplifies your calculations.

Derivatives of Trigonometric Functions

Now we can more accurately represent the differentiation formulas for trigonometric, exponential and logarithmic functions. Often these rules involve the use of the chain rule so it is more helpful to be familiar with these versions over the more basic ones.

Theorem 9.3: Derivatives of Trigonometric Functions

Let $u = u(x)$ be a function. Then,

$$\frac{d}{dx}[\sin u] = (\cos u)u' \qquad \frac{d}{dx}[\sec u] = (\sec u \tan u)u'$$
$$\frac{d}{dx}[\cos u] = -(\sin u)u' \qquad \frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$$
$$\frac{d}{dx}[\tan u] = (\sec^2 u)u' \qquad \frac{d}{dx}[\cot u] = -(\csc^2 u)u'$$

Example 9.3: Differentiating Trigonometric Functions

a.) $y = x\sin(x)$

Solution. Here we need to apply the Product Rule

Since $y' = \dfrac{d}{dx}[fg] = f'g + fg'$

Let $f = x \quad , \quad g = \sin(x)$

$f' = 1 \quad , \quad g' = \cos(x)$

So our derivative is

$$y' = \sin(x) + x\cos(x)$$

b.) $y = \cos(3x)$

Solution. Here we need to apply the Chain Rule

Since $y' = \dfrac{d}{dx}[f(u)] = f'(u(x)) \cdot u'(x)$

Let $f(u) = \cos(u) \quad , \quad u(x) = 3x$

$f'(u) = -\sin(u) \quad , \quad u'(x) = 3$

So our derivative is

$$\begin{aligned} y' &= -\sin(u(x)) \cdot u'(x) \\ &= -\sin(3x)(3) \\ &= -3\sin(3x) \end{aligned}$$

Derivatives of the Exponential and Natural Logarithms

The last two rules covered here are derivatives of exponential function and the natural logarithm. Derivatives of logarithmic and exponential functions with bases other than e will be covered in a later lecture.

Theorem 9.4: Derivative of Exponential & Natural Logarithm

$$\frac{d}{dx}[e^u] = e^u u' \qquad \frac{d}{dx}[\ln(u)] = \frac{u'}{u}$$

Example 9.4: Derivative of Exponential Function

a.) Differentiate $y = e^{2x}$

Solution. Here we need to apply the Chain Rule

$$\text{Since} \quad y' = \frac{d}{dx}[f(u)] = f'(u(x)) \cdot u'(x)$$
$$\text{Let} \quad f(u) = e^u \quad , \quad u(x) = 2x$$
$$f'(u) = e^u \quad , \quad u'(x) = 2$$

So our derivative is $y' = 2e^{2x}$

b.) Differentiate $y = \ln(7x)$

Solution. Here we need to apply the Chain Rule. This can look a little different because of the nature of the derivative of the natural logarithm.

$$\text{Since} \quad \frac{d}{dx}[\ln(u)] = \frac{u'(x)}{u(x)}$$

$$\text{Let} \quad u(x) = 7x$$
$$u'(x) = 7$$

So our derivative is

$$y' = \frac{u'(x)}{u(x)} = \frac{7}{7x} = \frac{1}{x}$$

Why always the Chain Rule?

For each of these formulas if $u = x$ then $u' = 1$. So, for example

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

Your textbook first presents the basic versions of these rules and then later shows them using the chain rule. I have found that presenting them like that makes it seem like you need to know a whole new set of rules. Presenting these rules using *only* the chain rule helps your brain remember that you are always technically using the Chain Rule when differentiating transcendental functions.

Derivatives of Functions Involving Multiple Rules

There is a reason why derivatives have been arranged in a particular way in these notes. First, it keeps things organized, and second it makes it easy to calculate derivatives when multiple rules are involved. This process is demonstrated in the next example.

I recommend using this or a process similar to this. Not only does it result in less errors but it makes it easy to find and correct any errors you may have made along the way. Not to mention it makes it much easier to award partial credit on assignments and exams!

Example 9.5: A Multi-step Derivative

Differentiate

$$y = \frac{x^3}{(x^2+2)^2}$$

Solution. For no particular reason other than to demonstrate it can be done, we rewrite our function so we can use the Product Rule instead of the Quotient Rule

$$y = \frac{x^3}{(x^2+2)^2} = x^3(x^2+2)^{-2}$$

To find the derivative we have a Product Rule and in carrying out that Product Rule we will also have to use the Chain Rule.

Since $\quad y' = \frac{d}{dx}[fg] = f'g + fg'$

$$\begin{aligned} \text{Let} \quad & f = x^3 \quad , \quad g = (x^2+2)^{-2} \\ & f' = 3x^2 \quad , \quad g' = \underbrace{\frac{d}{dx}[q(u)]}_{\Downarrow} = q'(u)u' = -2(x^2+2)^{-3}\cdot 2x \\ & \qquad\qquad\quad q = u^{-2} \quad , \quad u = x^2+2 \\ & \qquad\qquad\quad q' = -2u^{-3} \quad , \quad u' = 2x \end{aligned}$$

So our derivative is

$$\begin{aligned} y' &= 3x^2(x^2+2)^{-2} + x^3(-4x(x^2+2)^{-3}) \\ &= 3x^2(x^2+2)^{-2} - 4x^4(x^2+2)^{-3} \\ &= \frac{3x^2}{(x^2+2)^2} - \frac{4x^4}{(x^2+2)^3} \end{aligned}$$

Now in finding g' I used the letters q and u to define the functions used in the application of the chain rule at that step. I did this only so that my derivative pieces didn't get confused. Any letters can be used as long as you clearly label which is which. Writing out the general rule each time you do a derivative not only shows someone what you're doing but also makes these rules easier to remember.

Lecture 10

Rates of Change

Sections
3.1.5-3.1.6; 3.4

Introduction

Up until now, we have seen the derivative interpreted as the slope of the tangent line. Another common use of the derivative is as a rate of change. That is, the rate of change of one variable with respect to another variable, usually time. Rates of change are seen everywhere in the real world. In almost every field we have "stuff" that happens over time. Whether that is a chemical reaction, distance an object travels, growth of a population, etc. We can track changes happening over time much more easily than we can record any other kind of data. This means that in real world applications we often actually have information on the derivative and not the function itself! More about this idea comes toward the end of the course and is studied much more in depth in an ordinary differential equations course.

Average vs Instantaneous Rate of Change

Since we only know how to take derivatives at this point the examples in this lecture will start with the function itself and then determine a rate of change. A key piece of notation you will see throughout this section is Δ which means "change in". For example,

$$\Delta t = t_2 - t_1$$

would be the change in time. This notation is seen in the definition below where we begin with a discussion of the average rate of change of a function.

Definition 10.1: Average Rate of Change

The **average rate of change** in f between $t = a$ and $t = a + \Delta t$ is the slope of the corresponding secant line between $(a, f(a))$ and $(a + \Delta t, f(a + \Delta t))$ given by

$$m_{av} = \frac{f(a + \Delta t) - f(a)}{a + \Delta t - a} = \frac{f(a + \Delta t) - f(a)}{\Delta t}$$

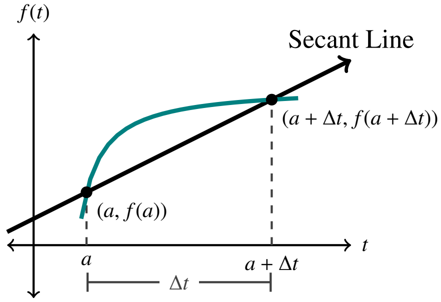


Fig. 10.1: Slope of the secant line

The average rate of change can graphically be described as the slope of the secant line. The rate of change at a single point is known as the **instantaneous** rate of change.

How do we find the rate of change at exactly one point? To find the instantaneous rate of change we want the distance Δt to become as small as possible. In other words, we want to obtain the tangent line.

You should recognize that this is exactly how we began to define the definition of the derivative. This is the same definition we used before only instead of h we call this distance Δt.

Definition 10.2: Instantaneous Rate of Change

The **instantaneous rate of change** of the function $f(t)$ at a is the slope of the tangent line given by

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(a + \Delta t) - f(a)}{\Delta t}$$

Objects in Motion

A common example we see regarding rates of change has to do with the model of an object in motion. We will begin by providing some basic definitions and then demonstrate their use through some examples. We start with a position function which lets us know where a particle may be at a given time t. In many textbooks the position function is denoted by $s(t)$ but any notation is acceptable.

Definition 10.3: Position and Displacement of an Object

- A function $s(t)$ that gives the position of an object as function of time t is known as the **position function**.
- The **displacement** of an object between a time t and a time Δt is given by

$$\Delta s = s(t + \Delta t) - s(t)$$

In the real world, we often do not have the position function. Think about when you drive in a car. Our speedometer gives us information on our positive velocity which is how fast we are traveling in the forward direction. In the United States, we record the speed at which a car is traveling as miles per hour (mi/hr). In other words, the distance traveled over the time we have been traveled. This is just a rate of change or in other words, the derivative. In the following examples we will be given a position function and use that to determine velocity function. Extending this idea again, we can find the rate of change of the velocity, or in other words, the acceleration function. These are summarized in the following definition.

Definition 10.4: Velocity and Acceleration

- The instantaneous rate of change of the position function is known as the **velocity function**, $v(t)$, and so

$$v(t) = \frac{d}{dt}[s(t)] = s'(t)$$

- The instantaneous rate of change of the velocity function is given by the **acceleration function**, $a(t)$, and so

$$a(t) = v'(t) = \frac{d}{dt}[s'(t)] = s''(t)$$

Example 10.1: Finding Velocity of an Object in Motion

The position of a particle with respect to time t in seconds (s) is given in meters (m) by the function

$$s(t) = t^3 - 12t^2 + 36t$$

Determine the following.

a.) Find the velocity function.

Solution. The velocity function is found by taking the derivative of the position function. In other words, $v(t) = s'(t)$ so we have

$$v(t) = s'(t) = 3t^2 - 24t + 36$$

b.) Find the particle's velocity after 3 seconds.

Solution. Since $v(t) = 3t^2 - 24t + 36$ then

$$v(3) = 3(3)^2 - 24(3) + 36 = -9\ \mathrm{m/s}$$

Example 10.2: Finding Acceleration of a an Object in Motion

Consider the same particle in Example 10.1. Find the following.

a.) The acceleration function.

Solution. The acceleration function is found by taking the derivative of the velocity function. In other words, $a(t) = v'(t) = s''(t)$. Since $v(t) = 3t^2 - 24t + 36$ we have

$$a(t) = v'(t) = 6t - 24$$

b.) The particle's acceleration after 3 seconds.

Solution. Since $a(t) = 6t - 24$ then

$$a(3) = 6(3) - 24 = -6\ \mathrm{m/s^2}$$

The direction in which the particle is moving at any time t is found by using the velocity function. The three possible cases are covered in the following definition.

Definition 10.5: Direction in Which an Object Travels

Suppose particle is moving with velocity given by $v(t)$. Then when

$v(t) > 0$	the particle is moving in the **positive direction**.
$v(t) = 0$	the particle is **at rest**.
$v(t) < 0$	the particle is moving in the **negative direction**

It may seem trivial to calculate the distance traveled by an object. However, we often will not be talking about objects traveling in a straight line and we will deal with objects that are not always traveling forward. To calculate total distance then we need to consider intervals where an object is moving forward *and* moving backward.

Definition 10.6: Distance Traveled by an Object

The **total distance** traveled by an object on an interval from $t = t_1$ to $t = t_2$ is given by

$$|s(t_2) - s(t_1)|$$

Example 10.3: Determining Distance Traveled by an Object in Motion

Consider the same particle in Example 10.1. Find the following.

a.) When the particle is at rest.

Solution. The particle is at rest when $v(t) = 0$. This occurs when

$$3t^2 - 24t + 36 = 3(t-2)(t-6) = 0$$

and so the particle is at rest when $t = 0$, $t = 2$, and $t = 6$. A few things to note here. $t = 0$ is a solution even though it is not a root of the polynomial because there is no such thing as negative time, and so negative t values should be excluded (meaning the particle has not started moving yet at time 0). Also, since $v(t)$ is zero at these points, these are also the points where the velocity function may change sign.

b.) When the particle is moving in the positive and negative direction.

Solution. Since the particle is at rest at $t = 2$ and $t = 6$ we will test the intervals

$$0 < t < 2, \quad 2 < t < 6, \quad t > 6$$

to determine the sign of the velocity function $v(t)$. We evaluate $v(t)$ at $t = 1, t = 3, t = 7$.

$$\begin{aligned} v(1) &= 3\Big((1)-2\Big)\Big((1)-6\Big) = 15 & \qquad v(t) > 0 \\ v(3) &= 3\Big((3)-2\Big)\Big((3)-6\Big) = -9 & \qquad v(t) < 0 \\ v(7) &= 3\Big((7)-2\Big)\Big((7)-6\Big) = 15 & \qquad v(t) > 0 \end{aligned}$$

We see that $v(t)$ is moving in the positive direction on the intervals $0 \le t \le 2$ and $t \ge 6$. The particle is moving in the negative direction on $2 \le t \le 6$.

c.) The total distance traveled by the particle from $t = 0$ seconds to $t = 8$ seconds.

Solution. We find the following:

$$\begin{aligned} 0 \le t \le 2 &: \quad |s(2) - s(0)| = |32 - 0| = 32 \\ 2 \le t \le 6 &: \quad |s(6) - s(2)| = |0 - 32| = 32 \\ 6 \le t \le 8 &: \quad |s(8) - s(6)| = |32 - 0| = 32 \end{aligned}$$

And so the total distance traveled is $32 + 32 + 32 = 96$ meters.

Definition 10.7: Speed of an Object

The **speed** of an object is given by the absolute value of the velocity

$$\text{Speed} = |v(t)|$$

An object is **speeding up** when the velocity and acceleration have the same sign. In other words, when

$$v(t) > 0 \text{ and } a(t) > 0 \quad \text{or} \quad v(t) < 0 \text{ and } a(t) < 0$$

An object is **slowing down** when the velocity and acceleration have different signs. In other words, when

$$v(t) > 0 \text{ and } a(t) < 0 \quad \text{or} \quad v(t) < 0 \text{ and } a(t) > 0$$

Example 10.4: Determining Speed of an Object in Motion

In Figure 10.2 we have graphs of the position, velocity and acceleration functions of the particle in Example 10.1. Find the following.

a.) When the particle is speeding up.

b.) When the particle is slowing down.

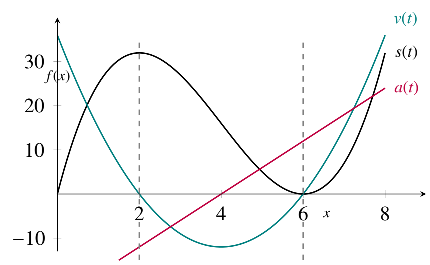


Fig. 10.2: Graphs of $s(t)$, $v(t)$, $a(t)$

Solution. From the graph we can see that $a(t) < 0$ on the interval $0 < t < 4$ and $a(t) > 0$ on the interval $4 < t < 8$. To compare the signs of $v(t)$ and $a(t)$ we can look at the graph but it may be easier to organize our information into a "sign chart" to better compare $v(t)$ and $a(t)$. You can use the information you found in Example 10.3 for $v(t)$ and carry out a similar procedure for $a(t)$ and then compare the intervals.

Interval (t)	$0 \le t \le 2$	$2 \le t \le 4$	$4 \le t \le 6$	$6 \le t \le 8$:
Sign of $v(t)$	+	−	−	+
Sign of $a(t)$	−	−	+	+

a.) To determine when the particle is speeding up we need to find when $v(t)$ and $a(t)$ have the same sign. We can see that this occurs on the intervals $0 \le t \le 4$ and $6 \le t \le 8$.

Thus, the particle is speeding up on the intervals $2 < t < 4$ and $6 < t < 8$.

b.) From the table we also observe that $a(t)$ and $v(t)$ have differing signs on the intervals $0 < t < 2$ and $4 < t < 6$.

Thus, the particle is slowing down on the intervals $0 < t < 2$ and $4 < t < 6$.

When using graphs always use technology to ensure that your graphs are accurate.

Rates of Growth

Many of the applications involving rates of change have to do with rates of *growth*. The same interpretations of average rate of change and instantaneous rate of change apply here. We demonstrate this through a simple example involving population growth.

Example 10.5: Determining Growth Rates of a Population

The population in Georgia (in thousands) from 1995 to 2005 is modeled by the function

$$p(t) = -0.27t^2 + 101t + 7055$$

Determine the following

a.) Determine the average growth rate of the population from 1995 to 2005.

Solution. The average rate of change of this function is given by the slope of the line between t_1 and t_2.

$$\frac{\Delta p}{\Delta t} = \frac{p(t_2) - p(t_1)}{t_2 - t_1}$$

We are asked to find the average growth rate from 1995 to 2005. Note that this model only gives the population between these two years. So we have that for 1995, $t = 0$ and for 2005, $t = 10$. So the average growth rate is

$$\frac{\Delta p}{\Delta t} = \frac{p(t_2) - p(t_1)}{t_2 - t_1} = \frac{p(10) - p(0)}{10 - 0} = \frac{8038 - 7055}{10} = 98.3 \text{ thousand people/year}$$

b.) Determine the growth rate of the population in 1997.

Solution. The growth rate in any single year will be given by the instantaneous growth rate. We know that this is given by the derivative $p'(t)$ which we find to be

$$p'(t) = -0.54t + 101$$

In 1997 we have $t = 2$ and so the growth rate in 1997 is

$$p'(2) = -0.54(2) + 101 = 99.2 \text{ thousand people/year}$$

c.) Determine the growth rate in 2005.

Solution. From part (b) we know that the derivative of $p(t)$ is

$$p'(t) = -0.54t + 101$$

In 2005 we have $t = 10$ and so the growth rate in 2005 is

$$p'(10) = -0.54(10) + 101 = 95.6 \text{ thousand people/year}$$

There are, of course, many other examples of the derivative's use in determining rates of change in a variety of areas. It is worth working through some of the other examples covered in the exercises in your textbook. This will give you a better idea of rates of change in your particular discipline.

Lecture 11

Implicit Differentiation Derivatives of Inverse Functions

Sections
3.7; 3.8

Implicit vs. Explicit Functions

So far we have been taking derivatives of explicit functions. In other words, functions like

$$f(x) = \sqrt{x+2}, \quad g(x) = 3x^2 + 7x + 9, \quad h(x) = 3, \text{ etc.}$$

A function can also be implied by an equation. These types of functions are known as *implicit functions*. These are functions like

$$y = \sqrt{x+y}, \quad xy = \sin(y^2) + 2x + 6y, \quad x^2 + y^2 = 1, \text{ etc.}$$

In other words, we have an equation where we see y inside another function or on both sides of our equation. An implicit function can sometimes be written explicitly like

$$x^2 + y^2 = 1 \implies y = \pm\sqrt{1-x^2}$$

The issue with doing this is that now we need to be careful about our domain. Often implicit functions *cannot* be written as an explicit function. An example of this is the function

$$y = \sqrt{x+y}$$

There is no way to get it to be a function that is y on one side and terms only involving the variable x on the other. For the equation just given we will always have a y term on both sides of the equation. A special type of differentiation allows us to find the derivatives of these implicit functions. The process is referred to as *implicit differentiation*. It allows us to completely avoid writing a function explicitly.

The Implicit Differentiation Process

Here is one of those times where it may be helpful to use the $\frac{dy}{dx}$ notation. More specifically, recall that this notation is saying "differentiate the function y with respect to x". In this course we are only dealing with *single variable* calculus. This means that y will be treated as a function and not as a variable. In other words, $y = y(x)$. This is different from treating y as if it were a variable. These types of functions are handled in multivariable calculus where you see how to handle functions with multiple variables. So for this course remember, ***y* is function of *x***.

Example 11.1: Differentiate with Respect to x

Differentiate the following.

a.) $\frac{d}{dx}[x^4]$

Solution. Note that here the variable of the term we are differentiating with respect to "matches" the variable in $\frac{d}{dx}$. In this case, we simply differentiate as we have before using the power rule we learned in the last lesson. So,

$$\frac{d}{dx}[x^4] = 4x^3$$

b.) $\frac{d}{dx}[y^4]$

Solution. Here our variables do not "match". This means we are differentiating an implicit function y, with respect to x. In other words, we are taking the derivative of a function that contains other functions and so we need to use the chain rule. Your textbook uses $\frac{dy}{dx}$ to represent $y'(x)$. Both notations are presented below.

Using $\frac{dy}{dx}$ notation:

$$\frac{d}{dx}[y^4] = 4y^3\frac{dy}{dx}$$

Using prime notation:

$$\frac{d}{dx}[y^4] = 4y^3y'$$

If you have trouble remembering to treat y as a function when using prime notation you can always write is as $y = y(x)$, i.e.

$$\frac{d}{dx}[y^4] = \frac{d}{dx}[(y(x))^4] = 4(y(x))^3 \cdot y'(x)$$

Remark: For the remainder of this section I will primarily use prime notation. Keep in mind that your text will use $\frac{dy}{dx}$ notation.

Implicit differentiation often involves using several differentiation rules. I recommend splitting up your equation into a left hand side (LHS) and right hand side (RHS). This way you can find the derivative of each side and then put the pieces back together to solve for y'. This process will be demonstrated in the next example.

Example 11.2: Implicit Differentiation

Differentiate $x^3 + y^3 = 9xy$.

Solution. Solving this function explicitly for y can be done but it is too complicated to do by hand. It is easier to use implicit differentiation. We begin by taking the derivative of each side.

$$\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[9xy]$$

Since implicit differentiation can be complicated we're going to split this equation into the left hand side (LHS) and the right hand side (RHS). Let

$$\text{LHS} = \frac{d}{dx}[x^3 + y^3] \qquad \text{and} \qquad \text{RHS} = \frac{d}{dx}[9xy]$$

<u>For LHS:</u>
Using our basic differentiation rules we're going to break the LHS up so that we can see exactly what is going on with each term.

$$\text{LHS} = \frac{d}{dx}[x^3 + y^3] = \underbrace{\frac{d}{dx}[x^3]}_{(1)} + \underbrace{\frac{d}{dx}[y^3]}_{(2)}$$

(1) This derivative is only in terms of x so we just apply the power rule.

$$\frac{d}{dx}[x^3] = 3x^2$$

(2) Remember that $y = y(x)$ so to take the derivative of (2) we need to use the chain rule. We have

$$\frac{d}{dx}[y^3] = \frac{d}{dx}[(y(x))^3] = 3(y(x))^2 \cdot y'(x) = 3y^2y'$$

So for LHS we have

$$\begin{aligned} \text{LHS} &= \frac{d}{dx}[x^3 + y^3] \\ &= \frac{d}{dx}[x^3] + \frac{d}{dx}[y^3] \\ &= 3x^2 + 3y^2y' \end{aligned}$$

<u>For RHS:</u>
Now let RHS $= \frac{d}{dx}[9xy]$. Note that this is a product of two functions! To take the derivative we need to use the product rule.

$$\text{RHS} = \frac{d}{dx}[9xy] = \frac{d}{dx}[fg] = f'g + fg'$$

$$\text{Let} \quad f = 9x \quad , \quad g = y$$
$$f' = 9 \quad , \quad g' = y'$$

So RHS $= \frac{d}{dx}[9xy] = 9y + 9xy'$.

Now we bring our LHS and RHS back together.

$$\frac{d}{dx}[x^3 + y^3] = \frac{d}{dx}[9xy]$$
$$\implies 3x^2 + 3y^2y' = 9y + 9xy'$$

We now need to solve for y'. To do this we collect all terms with y' together on one side and move remaining terms to the other side.

$$\begin{aligned}
3x^2 + 3y^2y' &= 9y + 9xy' && \\
3y^2y' - 9xy' &= 9y - 3x^2 && \text{Isolate } y' \text{ terms} \\
y'(3y^2 - 9x) &= 9y - 3x^2 && \text{Factor out a } y' \\
y' &= \frac{9y - 3x^2}{3y^2 - 9x} && \text{Solve for } y' \\
&= \frac{3y - x^2}{y^2 - 3x} && \text{Simplifying}
\end{aligned}$$

Note that in this example our derivative is also an implicit function. This may or may not be the case when carrying out implicit differentiation.

Remember that it is important to write out all of your steps. This may seem like a lot of work just to take a derivative. Be warned that the most common mistake students make is to forget to use the chain rule when taking the derivative of a function involving y. The more you write out your work the easier it will be to identify errors you have made and quickly correct them. When first practicing these problems do not skip any steps and take up a minimum of a half page if not more!

Example 11.3: Implicit Differentiation with Trigonometric Functions

Differentiate $x + y = \cos(y)$.

Solution. This is an example of a function that cannot be solved explicitly for y. So again we will need to use implicit differentiation. Taking the derivative of both sides we have

$$\frac{d}{dx}[x + y] = \frac{d}{dx}[\cos(y)]$$

The LHS is simple in this case. Let

$$\text{LHS} = \frac{d}{dx}[x + y] = \frac{d}{dx}[x] + \frac{d}{dx}[y] = 1 + y'$$

Now for the RHS we will need to apply the chain rule. So

$$\text{RHS} = \frac{d}{dx}[\cos(y)] = -\sin(y)y'$$

Now we bring our LHS and RHS back together

$$\frac{d}{dx}[x + y] = \frac{d}{dx}[\cos(y)] \implies 1 + y' = -\sin(y)y'$$

Now solve for y'

$$\begin{aligned} y' + \sin(y)y' &= -1 && \text{Isolate } y' \text{ terms} \\ y'(1+\sin(y)) &= -1 && \text{Factor our } y' \\ y' &= \frac{-1}{1+\sin(y)} && \text{Solve for } y' \end{aligned}$$

Tangent Lines of Implicit Functions

We can also find lines tangent to graphs of implicit functions.

Example 11.4: Finding the Tangent Line of an Implicit Function

Find the tangent line of $x^2 + xy = 3 - y^2$ at point $(1, 1)$.

Solution. First we find the derivative y' using implicit differentiation. Taking the derivative of both sides with respect to x:

$$\frac{d}{dx}[x^2 + xy] = \frac{d}{dx}[3 - y^2]$$

Let LHS $= \dfrac{d}{dx}[x^2 + xy] = \underbrace{\dfrac{d}{dx}[x^2]}_{(1)} + \underbrace{\dfrac{d}{dx}[xy]}_{(2)}$.

(1) $\dfrac{d}{dx}[x^2] = 2x$

(2) $\dfrac{d}{dx}[xy] = \dfrac{d}{dx}[fg] = f'g + fg'$

$$\text{Let} \quad \begin{array}{ll} f = x \quad , & g = y \\ f' = 1 \quad , & g' = y' \end{array} \qquad \Longrightarrow \quad \frac{d}{dx}[xy] = y + xy'$$

So LHS $= \frac{d}{dx}[x^2 + xy] = \frac{d}{dx}[x^2] + \frac{d}{dx}[xy] = 2x + y + xy'$

Now for the RHS $= \frac{d}{dx}[3 - y^2] = \underbrace{\dfrac{d}{dx}[3]}_{(3)} - \underbrace{\dfrac{d}{dx}[y^2]}_{(4)}$.

(3) $\dfrac{d}{dx}[3] = 0$

(4) $\dfrac{d}{dx}[y^2] = \dfrac{d}{dx}[(y(x))^2] = 2y(x) \cdot y'(x) = 2yy'$

So RHS $= \frac{d}{dx}[3 - y^2] = \frac{d}{dx}[3] - \frac{d}{dx}[y^2] = 0 - 2yy' = -2yy'$.

Now we bring our LHS and RHS back together

$$2x + y + xy' = -2yy'$$

and solve for y'

$$\begin{aligned} xy' + 2yy' &= -y - 2x && \text{Isolate } y' \text{ terms} \\ y'(x + 2y) &= -y - 2x && \text{Factor our } y' \\ y' &= -\frac{y + 2x}{x + 2y} && \text{Solve for } y' \end{aligned}$$

The slope of our tangent line at the point (1, 1) is

$$y' = -\frac{(1) + 2(1)}{(1) + 2(1)} = -1$$

Equation of the line is then

$$y - 1 = -(x - 1) \implies y = -x + 2$$

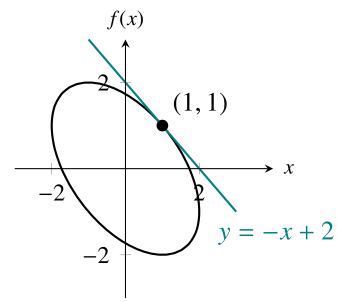


Fig. 11.1: Graph of $x^2 + xy = 3 - y^2$ and $y = -x + 2$

Higher Order Implicit Differentiation

Just as with explicit derivatives, we can also find higher order derivatives of implicit functions.

Example 11.5: Finding the Second Derivative of an Implicit Function

Find the second derivative of $x^2 + y^2 = 9$.

Solution. To find the second derivative we first need to find the first derivative.
First Derivative y':
We differentiate each side with respect to x

$$\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[9]$$

Now on the RHS we just have the derivative of a constant which we know is 0. On the LHS we have

$$\text{LHS} = \frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = 2x + 2y \cdot y'$$

Putting these pieces together we have

$$2x + 2y \cdot y' = 0$$

Solving for y' we have

$$\begin{aligned} 2yy' &= -2x \\ y' &= \frac{-\cancel{2}x}{\cancel{2}y} = -\frac{x}{y} \end{aligned}$$

So our first derivative is $y' = -\dfrac{x}{y}$.

Second Derivative y':

Recall that the second derivative $y''(x) = \dfrac{d}{dx}[y'(x)]$ so we take the derivative of each side of y'.

$$\frac{d}{dx}[y'] = \frac{d}{dx}\left[-\frac{x}{y}\right]$$

On the LHS we just have y''. On the RHS we have

$$\frac{d}{dx}\left[-\frac{x}{y}\right] = \frac{d}{dx}\left[\frac{f}{g}\right] = \frac{f'g - fg'}{g^2}$$

$$\text{Let} \quad \begin{array}{lcl} f = -x & , & g = y \\ f' = -1 & , & g = y' \end{array} \qquad \Longrightarrow \frac{d}{dx}\left[-\frac{x}{y}\right] = \frac{-y + xy'}{y^2}$$

So putting these pieces back together our second derivative is $y''(x) = \dfrac{-y + xy'}{y^2}$

Derivatives of Inverse Trigonometric Functions

The derivative formulas for the inverse trig functions are obtained using implicit differentiation. The derivation of these formulas is omitted here but we summarize these rules below.

Theorem 11.1: Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x then

$$\frac{d}{dx}[\sin^{-1}(u)] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\csc^{-1}(u)] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}[\cos^{-1}(u)] = \frac{-u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\sec^{-1}(u)] = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}[\tan^{-1}(u)] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\cot^{-1}(u)] = \frac{-u'}{1+u^2}$$

Lecture 12

Logarithmic Differentiation

Section
3.9.2-3.9.3

Derivatives of Logarithmic Functions

We have already seen the derivative of the natural exponential and logarithmic functions. We will now include the rules for these functions of base b. Please refer to your textbook for the derivation of these rules. We simply summarize them below.

Theorem 12.1: Derivatives of Exponential & Logarithmic Functions

- Derivatives of Exponential Functions

$$\frac{d}{dx}[e^u] = e^u u' \qquad \frac{d}{dx}[b^u] = b^u \ln(b) u'$$

- Derivatives of Logarithmic Functions

$$\frac{d}{dx}[\ln(u)] = \frac{u'}{u} \qquad \frac{d}{dx}[\log_b(u)] = \frac{u'}{\ln(b) \cdot u}$$

where $u = u(x)$ is a differentiable function of x.

We will demonstrate use of these derivatives in the following examples.

Example 12.1: Differentiating Logarithmic Functions

Differentiate the following functions.

a.) $f(x) = \ln(5x)$

Solution.

$$f'(x) = \frac{d}{dx}[\ln(5x)] = \frac{\frac{d}{dx}[5x]}{5x} = \frac{\cancel{5}}{\cancel{5}x} = \frac{1}{x}$$

b.) $g(x) = \ln(\cos(x))$

Solution.

$$g'(x) = \frac{d}{dx}[\ln(\cos(x))] = \frac{\frac{d}{dx}[\cos(x)]}{\cos(x)} = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$$

Now that we have seen how to take derivatives of logarithmic functions we can now use properties of logarithms to simplify the process of taking derivatives of complicated functions. This is known as *logarithmic differentiation*. You can technically use a logarithm with any base in this process but in mathematics we almost always use the natural logarithm. The three main properties you should remember are listed in the following definition.

Definition 12.1: Properties of Natural Logarithms

Product Rule: $\ln(xy) = \ln(x) + \ln(y)$

Quotient Rule: $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

Power Rule: $\ln(x^r) = r \cdot \ln(x)$, where r is a real number

Note that these are the *only* rules allowed! Many students make mistakes applying these rules. Remember that the logarithm is just a way of differently representing a function with an exponent. This means that the operations we carry out for exponents are the same ones for logarithms.

Logarithmic Differentiation

Our goal in logarithmic differentiation is to apply properties of logarithms to break up the given function into smaller pieces that are easier to differentiate. This may seem like more work than is needed but it can actually simplify your work in a lot of cases.

Example 12.2: Using Logarithmic Differentiation

Differentiate the function

$$y = x^2\sqrt{x^2+1}$$

Solution. First, we take the natural log (ln) of both sides.

$$\begin{aligned}
\ln(y) &= \ln(x^2\sqrt{x^2+1}) \\
&= \ln(x^2(x^2+1)^{\frac{1}{2}}) && \text{Rewrite with exponents} \\
&= \ln(x^2) + \ln((x^2+1)^{\frac{1}{2}}) && \text{Apply Product Rule of Logarithms} \\
&= 2\ln(x) + \frac{1}{2}\ln(x^2+1) && \text{Apply Power Rule of Logarithms}
\end{aligned}$$

Now we have the implicit function

$$\ln(y) = 2\ln(x) + \frac{1}{2}\ln(x^2+1)$$

So we use implicit differentiation

$$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}[2\ln(x) + \frac{1}{2}\ln(x^2+1)]$$

where

$$\text{LHS} = \frac{d}{dx}[\ln(y)] = \frac{y'}{y}$$

and

$$\text{RHS} = \frac{d}{dx}\left[2\ln(x) + \frac{1}{2}\ln(x^2+1)\right] = 2\frac{d}{dx}[\ln(x)] + \frac{d}{dx}\left[\frac{1}{2}\ln(x^2+1)\right]$$
$$= 2\left(\frac{1}{x}\right) + \frac{1}{2}\left[\frac{\frac{d}{dx}[x^2+1]}{x^2+1}\right] = \frac{2}{x} + \frac{1}{2}\left[\frac{2x}{x^2+1}\right] = \frac{2}{x} + \frac{x}{x^2+1}$$

Now we bring our LHS and RHS back together we have

$$\frac{y'}{y} = \frac{2}{x} + \frac{x}{x^2+1}$$

and solving for y' we have

$$y' = y\left(\frac{2}{x} + \frac{x}{x^2+1}\right)$$

Unlike previous examples we have seen in this case we actually started with an explicit function $y = x^2\sqrt{x^2+1}$. So, we must substitute this into the derivative function.

$$y' = y\left(\frac{2}{x} + \frac{x}{x^2+1}\right) = x^2\sqrt{x^2+1}\left(\frac{2}{x} + \frac{x}{x^2+1}\right)$$

Of course we could have found this derivative using the product and chain rules of differentiation and not bothered with using the logarithm. Try calculating the derivative this way and compare your result with the one above.

You may notice that I never use the formula for the derivative of $\log_b(x)$. This is because I can never remember it. Also, we don't really need it! We can always change the base of any logarithm to one we prefer to use. Recall the following definition from algebra.

Definition 12.2: Change of Base for Logarithms

To change from a base b to a base a we use

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

Note that since $\ln(x) = \log_e(x)$ we also have

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}$$

I'd rather remember fewer rules. You'll notice that by doing this you get exactly the same derivative as you would using the formula for base b!

Differentiating Functions of Form $y = f(x)^{g(x)}$

Logarithmic differentiation also gives us a means of differentiating complicated functions of the form $y = f(x)^{g(x)}$. These are sometimes called tower functions.

Example 12.3: Differentiation Functions of Form $y = f(x)^{g(x)}$

Differentiate $y = x^{\cos(x)}$

Solution. As before we take the natural log (ln) of both sides. This allows us to bring the function in our exponent "down" out of the exponent. We do this by taking advantage of the power rule of logarithms.

$$\ln(y) = \ln(x^{\cos(x)}) = \cos(x)\ln(x)$$

Now we have the implicit function

$$\ln(y) = \cos(x)\ln(x)$$

Using implicit differentiation

$$\frac{d}{dx}[\ln(y)] = \frac{d}{dx}[\cos(x)\ln(x)]$$

We see that again we have

$$\text{LHS} = \frac{d}{dx}[\ln(y)] = \frac{y'}{y}$$

$$\text{RHS} = \frac{d}{dx}[\cos(x)\ln(x)] = \frac{d}{dx}[fg] = f'g + fg'$$

$$\text{Let} \quad f = \cos(x) \quad , \quad g = \ln(x)$$

$$f' = -\sin(x) \quad , \quad g' = \frac{1}{x}$$

So RHS $= -\sin(x)\ln(x) + \cos(x)\left(\frac{1}{x}\right)$.

$$\frac{y'}{y} = \cos(x)\left(\frac{1}{x}\right) - \sin(x)\ln(x) \implies y' = y(\cos(x)\left(\frac{1}{x}\right) - \sin(x)\ln(x))$$

Again, we must plug $y = x^{\cos(x)}$ in for y

$$y' = x^{\cos(x)}\left(\cos(x)\left(\frac{1}{x}\right) - \sin(x)\ln(x)\right)$$

Of course you could use the chain rule and the formula for the derivative of b^x and achieve the same result but I find that it is more complicated and confusing that just using the logarithm rules. For either method you use, remember that it is very important to stay organized!

Lecture 13

Related Rates

Section
4.1

Introduction

In Lecture 11 we saw that the chain rule can be used to find a derivative implicitly. The chain rule can also be used to find rates of change of several related variables which change with respect to time. These types of problems are known as *related rates* problems.

It is very important to review geometry formulas and basic trigonometry before proceeding. Most of the difficulty in these problems lies in setting them up properly, from there the calculus is usually very straight forward.

Chain Rule in Leibniz Notation

Recall from Lesson 9 that the chain rule is defined as

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

We now show the chain rule using *Leibniz Notation*. You may find this notation helpful when working on these types of problems.

Definition 13.1: Chain Rule in Leibniz Notation

Let $f = f(u(x))$ and $u = u(x)$ both be differentiable functions. Then

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

For example, the volume of a sphere is given by the formula

$$V(r) = \frac{4}{3}\pi r^3$$

If the radius r changes over time then we can say that the radius is a function of time. In other words, $r(t)$. So really we have

$$V(r(t)) = \frac{4}{3}\pi(r(t))^3$$

Taking the derivative of this function will require that we use the chain rule. We obtain

$$V'(r(t)) = \frac{4}{3}\pi[3(r(t))^2] \cdot r'(t) = 4\pi(r(t))^2 \cdot r'(t)$$

In Leibniz notation we would write $r'(t) = \frac{dr}{dt}$ and so

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

This is one of the few times I find this notation helpful. Think of the d as meaning *difference*. In other words, $\frac{dr}{dt}$ is the *difference in radius* with respect to the *difference in time*. You can also think of d as meaning *change in*.

We now proceed by showing a few examples. Students often find that related rates are the most challenging since they can be difficult to set up and organize your work. For nearly all word problems like this, I use a process known as "GASCAP". This may be a method similar to something you already do or you may have seen something like it in another course. A separate handout will be made available which outlines this method for word problems.

Note that in the figures accompanying each problem the dotted lines with arrows indicate quantities that are changing and the "direction" in which they are changing. For shapes, solid lines indicate where an object "started" and the dashed lines indicate what it is changing to.

Examples of Related Rates

Example 13.1: Area of a Circle

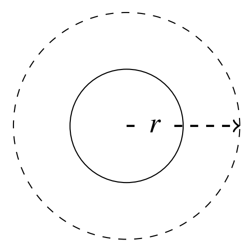
The radius r of a circle is increasing at a rate of $4\,\text{cm/min}$. Find the rates of change of the area when $r = 8\,\text{cm}$ and $r = 32\,\text{cm}$.

Solution.

Given:
We are given the rate at which the radius is changing: $\frac{dr}{dt} = 4\,\text{cm/min}$.

We also know that the area of a circle is given by $A = \pi r^2$.

Sketch:



Asked:
We need to find the rate the *area* is changing when $r = 8$ and $r = 32$. i.e.

$$\frac{dA}{dt} = ? \quad \text{when } r = 8, \text{ and } r = 32$$

Compute: First we differentiate $A = \pi r^2$ implicitly with respect to t. We get

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Since $\frac{dr}{dt} = 4$ then

$$\text{When } r = 8: \quad \frac{dA}{dt} = 2\pi(8)(4) = 64\pi\,\text{cm}^2\text{/min}$$

$$\text{when } r = 32: \quad \frac{dA}{dt} = 2\pi(32)(4) = 256\pi\,\text{cm}^2\text{/min}$$

Answer:
If the radius is increasing at a rate of $4\,\text{cm/min}$ then the area is increasing at a rate of $64\pi\,\text{cm}^2\text{/min}$ when $r = 8$ and at a rate of $256\pi\,\text{cm}^2\text{/min}$ when $r = 32$.

Example 13.2: Melting Snowball

If a snowball melts so that its surface area decreases at a rate of 1 $\mathrm{cm^2/min}$, find the rate at which the diameter decreases when the diameter is 10 cm.

Solution.

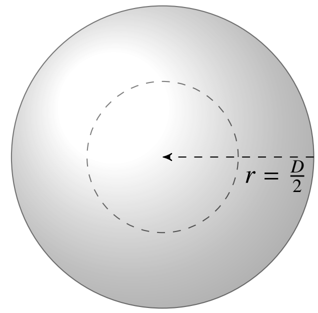
Given:

Surface Area: $S = 4\pi r^2$

Diameter: $D = 2r$

Rate surface area decreases: $\frac{dS}{dt} = -1\ \mathrm{cm^2/min}$

Sketch:



Asked:

We want to find the rate at which the diameter is decreasing when $D = 10$ cm. In other words,

$$\frac{dD}{dt} = ? \quad \text{when } D = 10$$

Compute:

Our equation for surface area is given in terms of the radius. Since $D = 2r$ this implies that the radius is given by $r = \frac{D}{2}$. Substituting this into our equation we obtain

$$S = 4\pi\left(\frac{D}{2}\right)^2 = 4\pi\frac{D^2}{4} = \pi D^2$$

Differentiating implicitly with respect to t we have

$$\frac{dS}{dt} = 2\pi D\frac{dD}{dt}$$

Solving for $\frac{dD}{dt}$ we have

$$\frac{dD}{dt} = \frac{1}{2\pi D}\frac{dS}{dt}$$

Since $\frac{dS}{dt} = -1$ when $D = 10$ then

$$\frac{dD}{dt} = \frac{1}{2\pi(10)}(-1) = -\frac{1}{20\pi}\ \mathrm{cm/min}$$

Answer:

When the surface area is decreasing at a rate of $-1\ \mathrm{cm^2/min}$ then the diameter is decreasing at a rate of $-\frac{1}{20\pi}\ \mathrm{cm/min}$.

For these next two problems it is vital to have an understanding of *similar triangles*. In short, when two triangles have two angles that are the same then we can express the lengths of the sides of one triangles as a ratio of the other. This will be demonstrated in the following example.

Example 13.3: Conical Tank

A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

Solution.

Given:

h :	height of water
r :	radius of water
diameter	= 10 ft
radius	= 5 ft
depth	= 12 ft
volume	$\frac{1}{3}\pi r^2 h$
Rate water flows in:	$\frac{dV}{dt} = 10\ \mathrm{ft}^3/\mathrm{min}$

Sketch:

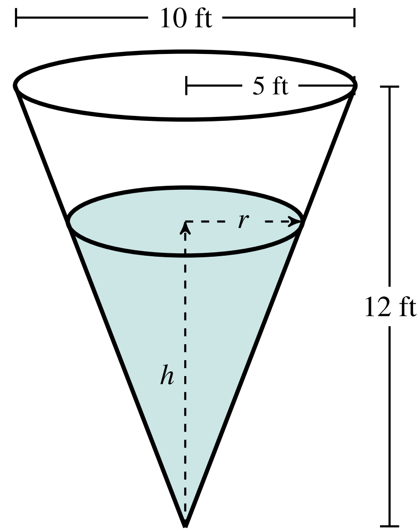


Fig. 13.1: Conical Tank

Asked:
Need to find the rate of change of the *depth of the water* when the height is $h = 8$ ft. i.e.

$$\frac{dh}{dt} = ? \quad \text{when } h = 8$$

Compute:
We know that volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$. Since we don't like to handle functions of two variables at this level and we are given a value for h we want to eliminate the r in our equation.

From our sketch of the given information we see that we can construct the triangle pictured in Figure 13.2. This will allow us to use the property of similar triangles to find an expression for r. We obtain

$$\frac{r}{h} = \frac{5}{12} \implies r = \frac{5}{12}h$$

Substituting this into our equation we have

$$V = \frac{1}{3}\pi\left(\frac{5}{12}h\right)^2 h = \frac{25}{432}\pi h^3$$

Now we have a function solely of the variable h. Differentiating implicitly with respect to t we obtain

$$\frac{d}{dt}V = \frac{25}{432}\pi\left(3h^2\frac{dh}{dt}\right) = \frac{75}{432}\pi h^2\frac{dh}{dt}$$

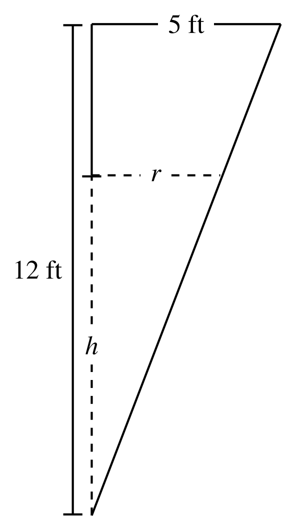


Fig. 13.2: Conical Tank Triangles

Now solving for $\frac{dh}{dt}$ we obtain

$$\frac{dh}{dt} = \frac{432}{75\pi h^2}\frac{dV}{dt}$$

We are given that $\frac{dV}{dt} = 10\,{}^{\text{ft}^3}\!/\!_{\text{min}}$ when $h = 8$ ft so

$$\frac{dh}{dt} = \frac{432}{75\pi(8)^2}(10) = \frac{9}{10\pi}\,{}^{\text{ft}}\!/\!_{\text{min}}$$

Answer:
When water is flowing into the tank at $10\,{}^{\text{ft}^3}\!/\!_{\text{min}}$ the height of the water in the tank is increasing at a rate of $\frac{9}{10\pi}\,{}^{\text{ft}}\!/\!_{\text{min}}$.

Example 13.4: Shadow Problem

A man 6 feet tall walks at a rate of 5 feet per second away from a light that is 15 feet above the ground (see figure). When he is 10 feet from the base of the light, at what rate is the tip of his shadow moving and at what rate is the length of his shadow changing?

Solution.

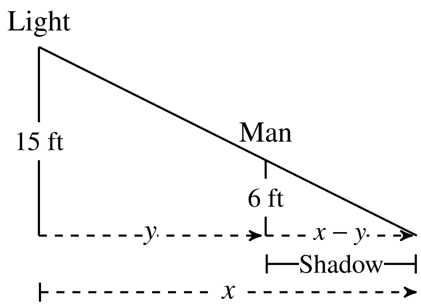
Given:

- x: distance from light to end of shadow
- y: distance from light to man
- $x - y$ shadow length
- 15 ft: height of light
- 6 ft: height of man

Rate distance to man changes:

$$\frac{dy}{dt} = 5\,{}^{\text{ft}}\!/\!_{\text{sec}}$$

Sketch:



Asked:
Rate at which the tip of the shadow is moving: $\frac{dx}{dt} =?$.
Rate at which length of the shadow is changing: $\frac{d}{dt}(x - y) =?$.

Compute:
Again, we must use the property of similar triangles.

$$\frac{15}{6} = \frac{x}{x - y}$$

Simplifying this equation we obtain

$$9x - 15y = 0$$

Differentiating implicitly with respect to t we obtain

$$\frac{d}{dt}(9x - 15y) = 0 \implies 9\frac{dx}{dt} - 15\frac{dy}{dt} = 0$$

Solving for $\frac{dx}{dt}$ we have

$$\frac{dx}{dt} = \frac{15}{9}\frac{dy}{dt}$$

Since $\frac{dy}{dt} = 5\,{}^{\text{ft}}\!/\!_{\text{sec}}$ then

$$\frac{dx}{dt} = \frac{15}{9}(5) = \frac{75}{9} = \frac{25}{3}\,{}^{\text{ft}}\!/\!_{\text{sec}}$$

Now that we have $\frac{dx}{dt}$ we can now find the rate at which the length of the shadow, $x - y$, is changing. We differentiate $x - y$ with respect to t and obtain

$$\frac{d}{dt}(x - y) = \frac{dx}{dt} - \frac{dy}{dt} = \frac{25}{3} - 5 = \frac{10}{3}\ \mathrm{ft/sec}$$

Answer:
When the distance from the light to the man changes at the rate 5 ft/sec the rate at which the tip of the shadow is moving is $\frac{25}{3}$ ft/sec and the length of the shadow changes at a rate of $\frac{10}{3}$ ft/sec.

Lecture 14

Linear Approximation and Differentials

Section
4.2

Linear Approximation

We know that the slope of a tangent line at a point $(a, f(a))$ is given by $f'(a)$. The general equation of a tangent line for any function $f(x)$ at $x = a$ can be defined as follows.

Definition 14.1: Equation of a Tangent Line at a Point

The **equation of the tangent line** at the point $(a, f(a))$ is given by

$$y = f(a) + f'(a)(x - a)$$

Let's take another look at the graph of some function f and its tangent line at $x = a$. If we were to walk along the curve of f we can see that points on the tangent line are very close to points on f as we get closer and closer to $x = a$ on either side.

This observation leads us to another key application of the derivative known as a **linear approximation** (sometimes called *linearization*.)

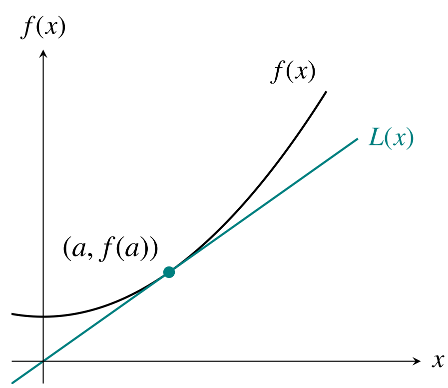


Fig. 14.1: Linear Approximation of f

Definition 14.2: Linear Approximation

The **linear approximation** of f near $x = a$ is given by $f(x) \approx L(x)$ where

$$L(x) = f(a) + f'(a)(x - a)$$

In other words, the derivative allows us to approximate the value of a function using a line. Why would we want to do this? Don't we have calculators that can evaluate functions for us so we can get exact values? While this is true, there are some functions that can be difficult to evaluate at certain points or may not be defined there. This allows us to approximate the value without evaluating the function itself.

Example 14.1: Using Linear Approximation

Use a linear approximation to estimate the value of $\sqrt{99.8}$.

Solution. To approximate the value $\sqrt{99.8}$ we recognize that this is the function $f(x) = \sqrt{x}$ evaluated at the point $x = 99.8$. We then choose $a = 100$ and find that $f(100) = \sqrt{100} = 10$ where our derivative is $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$.

Evaluating at $a = 100$ we have

$$f'(a) = f'(100) = \frac{1}{2}(100)^{-\frac{1}{2}}$$

So our linear approximation is the function

$$\begin{aligned} L(x) &= 10 + \frac{1}{20}(x - 100) \\ &= 10 + \frac{x}{20} - 5 \\ &= \frac{1}{20}x + 5 \end{aligned}$$

We evaluate $L(x)$ at $x = 99.8$ and obtain

$$L(x) = \frac{1}{20}(99.8) + 5 = 9.99$$

This is where $\approx$ versus $=$ is very important! $L(x)$ is the approximation of $f(x)$ which means $L(x) \approx f(x)$. Thus the linear approximation of $f(x)$ at $x = 99.8$ is

$$f(99.8) \approx L(99.8) = 9.99$$

In other words, $\sqrt{99.8} \approx 9.99$

Differentials

Consider again the equation of tangent line

$$y = f(a) + f'(a)(x - a)$$

when we use this to approximate the function f. The term $(x - a)$ is referred to as the **change in *x***, denoted by Δx. If this value is small then the **change in *y*** denoted by Δy is given as

$$\Delta y = f(a + \Delta x) - f(a)$$

This can be approximated by

$$\Delta y \approx f'(a)\Delta x$$

The quantity Δx is called the **differential of *x***, and is usually denoted as dx. Similarly, the quantity Δy is called the **differential of *y***, denoted dy.

Definition 14.3: Differentials

Let $y = f(x)$ be a differentiable function on an open interval I that contains x. The **differential of x**, denoted dx, is any non-zero real number and the **differential of y**, denoted dy, is given by

$$dy = f'(x)\,dx$$

In other words we have

$$\Delta y \approx dy \implies f(x+dx) - f(x) \approx dy = f'(x)\,dx$$

Example 14.2: Approximating Changes using Differentials

Approximate the change in the volume of a sphere when its radius changes from $r = 5$ ft to $r = 5.1$ ft.

Solution. We are given that r changes from $r = 5$ ft to $r = 5.1$ ft which means that

$$\Delta r = 5.1\,\text{ft} - 5\,\text{ft} = 0.1\,\text{ft}$$

Volume of a sphere is given by $V(r) = \frac{4}{3}\pi r^3$. By Definition 14.3 we have

$$\Delta V \approx V'(r)\,\Delta r = 4\pi r^2 \cdot \Delta r$$

We choose $r = 5$ and so along with $\Delta r = 0.1$ we have

$$\Delta V \approx 4\pi(5)^2 \cdot (0.1) = 10\pi \approx 31.416\,\text{ft}^3$$

So we find that a change of 0.1 ft in the radius results in a change in volume of the sphere of approximately $31.416\,\text{ft}^3$.

Calculating Error

Error in Approximation

Now our error associated with any approximation is in general just the difference between the actual function and our approximating function $L(x)$.

Definition 14.4: Error in Approximation

Let $L(x)$ be a function approximating the function $f(x)$. Then the **error in approximation** is given by

$$\text{Error} = f(x) - L(x)$$

Often, the direct error in approximation does not give a clear idea of the error in an approximation. In many applications it makes more sense to examine the **relative error**.

Definition 14.5: Relative Error

The **relative error** is given by

$$\text{Relative Error} = \frac{|\text{exact} - \text{approximate}|}{|\text{exact}|}$$

This can also be expressed as $\dfrac{\Delta f}{f} \approx \dfrac{df}{f}$

Multiplying the relative error by 100 gives the **percent error**

$$\text{Percent Error} = \frac{|\text{exact} - \text{approximate}|}{|\text{exact}|} \cdot 100$$

In Example 14.1 we found that $\sqrt{99.8} \approx 9.99$. Using Mathematica the value is given by 9.98999 which gives a relative error of 1.001×10^{-6}. This is a pretty good estimate considering we used a simple line to approximate this value! Note that different calculators and computer programs can sometimes output slightly different values as they may have different methods for rounding or truncating values.

Error Propagation

In real world situations there is a certain amount of error that can result from the tools used to take measurements. In this context x will represent the measured value of a variable and $x + \Delta x$ the actual value. Then Δx is the **error in measurement**.

Definition 14.6: Propagated Error

Let x be the measured value of a variable and $x + \Delta x$ the exact value. If x is used to compute $f(x)$ then the difference between $f(x + \Delta x)$ and $f(x)$ is the **propagated error** Δy:

$$\Delta y = f(x + \Delta x) - f(x)$$

In other words, $f(x + \Delta x)$ is the exact value of the function, $f(x)$ is the measured value and Δy represents the propagated error.

Example 14.3: Estimation of Error

The measured radius of a ball bearing is 0.7 in. If the measurement is correct to within 0.01 inch, estimate the propagated error in the volume V of the ball bearing.

Solution. The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere. Our measured radius is $r = 0.7$. The possible error is in the interval

$$-0.01 \leq \Delta r \leq 0.01$$

To approximate the propagated error in the volume, differentiate V to obtain $\frac{dV}{dr} = 4\pi r^2$ we want to approximate ΔV by dV so we have

$$\begin{aligned} \Delta V \approx dV &= 4\pi r^2 dr \\ &= 4\pi(0.7)^2(\pm 0.01) && \text{Substitute for } r \text{ and } dr \\ &\approx \pm 0.06158\,\text{in}^3 \end{aligned}$$

So, the volume has a propagated error of about 0.06 cubic inch.
The relative error is found by comparing V and dV. In other words we have the ratio

$$\begin{aligned} \frac{dV}{V} &= \frac{4\pi r^2\,dr}{\frac{4}{3}\pi r^3} = \frac{3\,dr}{r} \\ &= \frac{3}{0.7}(\pm 0.01) = \pm 0.0429 \end{aligned}$$

From this we have a percent error of 4.29%.

Differential Forms

Differentiation rules can be written in **differential form**. This is the form used to develop some of the integration rules you learn in the next semester of calculus. This comes up in more advanced courses but it is worth mentioning here.

Suppose u and v are differentiable functions of x. Then by the definition of differentials we have

$$du = u'\,dx \quad \text{and} \quad dv = v'\,dx$$

For example we can write the product rule as

$$d[uv] = \frac{d}{dx}[uv]\,dx = [vu' + uv']\,dx = u'v\,dx + uv'\,dx = u\,dv + v\,du$$

The same derivative formulas we learned before can be represented in this form.

Definition 14.7: Differential Formulas

Let u and v be differentiable functions of x.

Constant Multiple:	$d[cu] = c\,du$
Sum/Difference:	$d[u \pm v] = du \pm dv$
Product	$d[uv] = u\,dv + v\,du$
Quotient:	$d\left[\dfrac{u}{v}\right] = \dfrac{u\,dv - v\,du}{v^2}$

Lecture 15

Graphing Functions Using the First Derivative

Sections
3.2.2; 4.3; 4.4; 4.5.1-4.5.2

Introduction

The graph of a function can reveal important information about the function's behavior. Behavior of a function can include things like when it is increasing, decreasing, reaching a maximum or minimum, etc. We learn a variety of techniques in algebra which allow us to obtain this information.

Now that we know how to differentiate we can obtain this information faster by using the derivative. This lecture will cover how the derivative can be used.

Extrema of a Function

The *extrema* of a function refers to the maximum and minimum values a function may have. There are two main types of extrema that we will discuss here. When locating extreme values of a function we may want to talk about values of the function as a whole or over a smaller interval.

Recall that there are two types of intervals; *open* interval (a, b) that do not contain their endpoints, and *closed* intervals $[a, b]$ which do contain their endpoints. The type of interval we are using will determine what kind of maximum or minimum values we can locate.

First we will discuss what happens on a *closed* interval $[a, b]$.

Definition 15.1: Absolute Maximum and Minimum Values

Let f be defined on a *closed* interval $I = [a, b]$ containing c. Then the value $f(c)$ is the

(i) **absolute minimum of f on** I if $f(c) \leq f(x)$ for all x in I.

(ii) **absolute maximum of f on** I if $f(c) \geq f(x)$ for all x in I.

Do we always have maximum or minimum values of a function? On a closed interval the answer is yes!

Theorem 15.1: Extreme Value Theorem

If f is continuous on a *closed* interval $[a, b]$, then f has both a minimum and a maximum on the interval.

Of course we will not always be dealing with functions on closed intervals. For a function on an open interval (a, b) we refer to these extrema as *local* maximum or minimum.

Definition 15.2: Local Maximum and Minimum Values

Let f be a function defined on an *open* interval $I = (a, b)$ containing c. Then,

(i) $f(c)$ is the **local minimum of** f **on** I if $f(c) \leq f(x)$ for all x in I.

(ii) $f(c)$ is the **local maximum of** f **on** I if $f(c) \geq f(x)$ for all x in I.

This definition is demonstrated graphically in Figure 15.1.

On the open interval I_a the point $(a, f(a))$ is a local *maximum*. We can see that every value on either side of a is less than $f(a)$. In other words, $f(x) \geq f(a)$ for all points of the function in I_a.

On the interval I_b, the point $(b, f(b))$ is a local *minimum* value. We can see that every value on either side of b is greater than $f(b)$. In other words, $f(b) \leq f(x)$ for all points of the function in I_b.

These are called local minimum and maximum values as they are valid *locally* in the interval you have chosen. A local maximum or minimum value may not be a minimum or maximum at another point on your function!

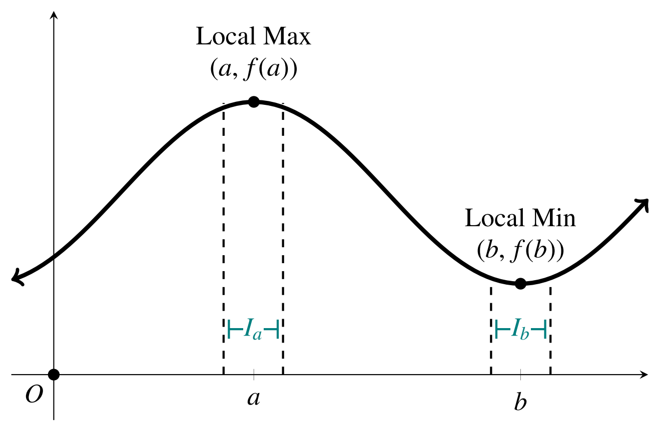


Fig. 15.1: Examples of Local Maximum and Local Minimum Values

Local minimum and maximum values are also referred to as **relative minimum** and **relative maximum** values as they are *relative* to the interval you are dealing with.

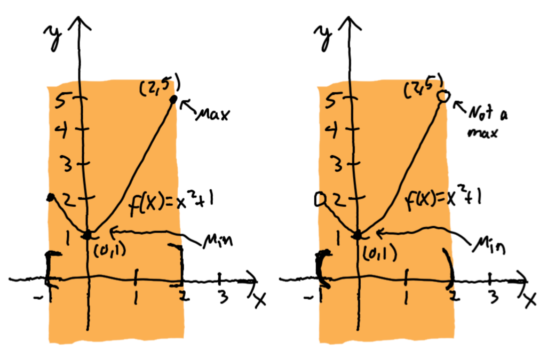


Fig. 15.2: Comparison of absolute extrema and local extrema

What is the difference between absolute and local extrema?

For absolute extrema we have restricted the interval to some finite set of points that includes the endpoints. We will always have a maximum and a minimum value in this case.

For local extrema we may not have a maximum or a minimum value on the interval chosen. See 15.2 for a graphical comparison of the two types of extrema.

Now we know how to find extrema if looking at the graph of a function. The main question now is how do we locate where these maximum and minimum values occur without a graph? The answer is found using the derivative of the function along with a key value known as a **critical point**.

The two types of critical points are represented graphically in Figure 15.3 and Figure 15.4

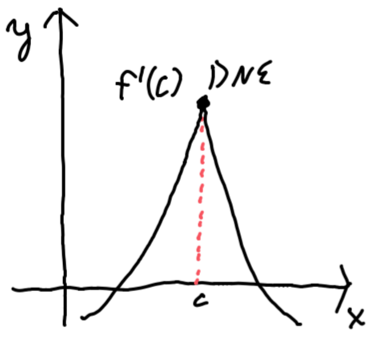


Fig. 15.3: When $f'(c)$ does not exist

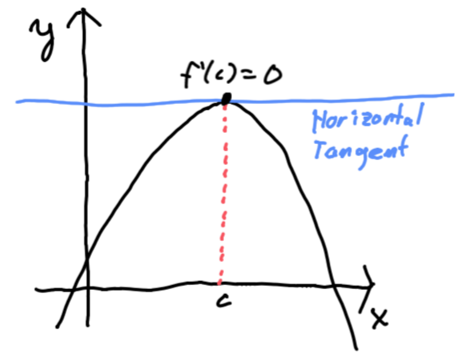


Fig. 15.4: When $f'(c) = 0$

Definition 15.3: Critical Point

A **critical point** of a function f is a point $x = c$ where $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem 15.2

If a function f has a local maximum or local minimum at $x = c$ then there is a critical point of f at c.

This theorem is telling us that local maximum and minimum values can occur *only* at critical points of f. The converse, (i.e. reverse) of this statement is not true. In other words, it is possible that a critical point is *not* a maximum or minimum value.

Sometimes a critical point is also called a *critical value*. This makes a bit more sense since the critical value tells us *where* a critical point is located (i.e. the x value of the point). In order to get the actual point itself, we would need to plug the value of x into our function to obtain the actual y value.

Ultimately, what this means is that for a *closed* interval $[a, b]$ we can follow these steps to find absolute max and min values of a function f.

Steps for finding absolute max and min values

(1) Find the critical points of f in (a, b).

(2) Evaluate f at each critical point.

(3) Evaluate f at a and b, the endpoints of the interval $[a, b]$.

(4) The least of these values is the absolute minimum, the greatest is the absolute max.

Recall that $x = c$ is actually the equation for a vertical line. Note that a critical point is located *at* $x = c$. Meaning that it is located on the line $x = c$. Saying that the critical point *is* $x = c$ is equivalent to saying that the the critical point is the entire line. Language matters!

Example 15.1: Locating Absolute Maximums and Minimums

Find the extrema of $f(x) = \frac{x^3}{3} - 4x$ on the interval $[-3, 3]$.

Solution. First we need the derivative f':

$$f'(x) = x^2 - 4 = (x-2)(x+2)$$

Note that the graph of $f(x)$ is depicted in Figure 15.5 for your reference only. You do not need to graph your results although you are encouraged to do so to check your work.

(1) Since $f'(x)$ is a polynomial it is continuous on $[-3, 3]$. There are no points where $f'(x)$ does not exist. So to find the critical points we set $f'(x) = 0$ and solve for x:

$$f'(x) = (x-2)(x+2) = 0$$

Thus, our critical points are at $x = -2$ and $x = 2$.

We are on a closed interval so we can determine an absolute maximum and minimum value by evaluating $f(x)$ at the critical points and at our endpoints $x = -3$, $x = 3$. We obtain the following values.

(2) Value of f at the critical points:

$$f(-2) = \frac{16}{3} \approx 5.666,$$
$$f(2) = \frac{-16}{3} \approx -5.666$$

(3) Value of f at the endpoints:

$$f(-3) = 3,$$
$$f(3) = -3$$

(4) The largest function value occurs at $x = -2$. This is the **absolute maximum**.

The least function value occurs at $x = 2$. This is the **absolute minimum**.

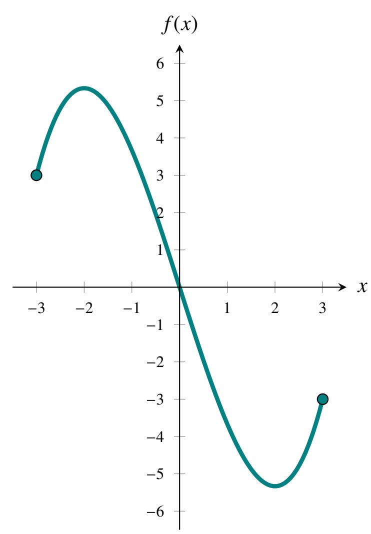


Fig. 15.5: Graph of $f(x)$

As you can see, a smaller critical point value does not always give you a minimum function value. Be very careful that you are evaluating the correct function at each of these steps. It can be easy to confuse the derivative and the original function. This is another example of why notation is very important.

Increasing and Decreasing Functions

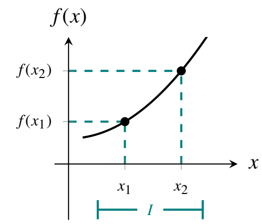
When is a function increasing or decreasing? Graphically, this means that as we move along the graph of a function from left to right if the graph is going up, it's increasing and if the graph is moving down the function is decreasing.

We start with a more formal definition of what it means for a function to be increasing or decreasing.

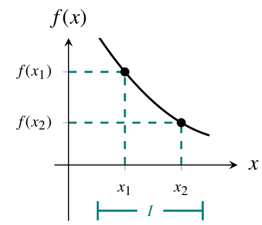
Definition 15.4: Increasing and Decreasing Functions

Let f be a function on an interval I then

f is **increasing** on I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I.



f is **decreasing** on I if $f(x_1) > f(x_2)$ for all $x_1 < x_2$ in I.



The first derivative gives us a means to find the intervals on which a function is increasing or decreasing.

Theorem 15.3: Test for Increasing/Decreasing Functions

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

(i) If $f'(x) > 0$ for all x in (a, b), then f is **increasing** on $[a, b]$.

(ii) If $f'(x) < 0$ for all x in (a, b), then f is **decreasing** on $[a, b]$.

(iii) If $f'(x) = 0$ for all x in (a, b), then f is **constant** on $[a, b]$.

Why is this true? Recall that the derivative is the slope of the tangent line. If $f'(x) > 0$ then the tangent line is pointing up and to the right. If $f'(x) < 0$ then the tangent line is pointing down and to the right. We can see this demonstrated in the figures below.

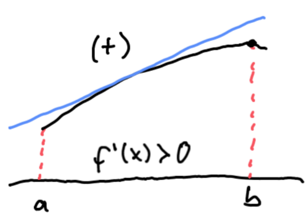


Fig. 15.6: f increasing when $f'(x) > 0$

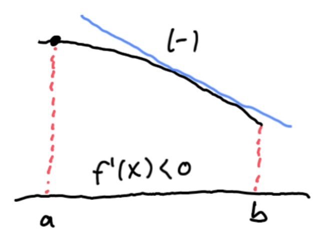


Fig. 15.7: f decreasing when $f'(x) < 0$

In order to find these intervals, recall what the Intermediate Value Theorem tells us: If $f(a) < 0$ and $f(b) > 0$ on an interval $[a, b]$ then there is a point where our function had to equal zero. Applying this to the derivative, we can establish intervals defined by the critical numbers of f i.e. when $f'(x) = 0$. We then test the sign of $f'(x)$ on that interval to determine the behavior.

Example 15.2: Determining Intervals of Increase and Decrease

Find the intervals on which $f(x) = \frac{x^3}{3} - 4x$ is increasing and decreasing.

Solution. From Example 15.1 we know that the critical numbers of f are $x = -2$ and $x = 2$. The easiest way to organize your tests is using a table:

Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
Test value (x)	$x = -3$	$x = 1$	$x = 3$
Value of $f'(x)$	$f'(-3) = 5$	$f'(1) = -3$	$f'(3) = 5$
Sign of $f'(x)$	$+$	$-$	$+$
Behavior	increasing	decreasing	increasing

So f is increasing on $(-\infty, -2)$ and $(2, \infty)$ and decreasing on $(-2, 2)$.

Note that when you are filling in your table, the actual value of $f'(x)$ doesn't really matter. We only really care about the sign.

The First Derivative Test

From our work in Example 15.1 we know that an absolute maximum occurs at $x = -2$. If we consider this function now on an *open* interval, say $(-\infty, \infty)$, then we know that at $x = -2$ we have a *local* maximum value. Likewise, we have a local minimum at $x = 2$.

Now think about what we saw in Example 15.2. From our test for increasing and decreasing we know that at the local maximum (located at $x = -2$) that f went from increasing to decreasing. We also saw that at the local minimum (located at $x = 2$) that f went from decreasing to increasing.

In other words, the first derivative gave us the means to determine whether a critical point gives us a local maximum or minimum value! This is known as the **first derivative test** and is formalized in the following theorem.

Theorem 15.4: First Derivative Test

Let c be a critical value of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval (except possibly at c) then $f(c)$ is classified as follows.

(i) If $f'(x)$ changes from negative to positive at c, then f has a *local minimum* at $(c, f(c))$.

(ii) If $f'(x)$ changes from positive to negative at c, then f has a *local maximum* at $(c, f(c))$.

(iii) If $f'(x)$ is positive on both sides of c or negative on both sides of c, then $f(c)$ is neither a relative minimum nor a relative maximum.

We can see this demonstrated in the figures below.

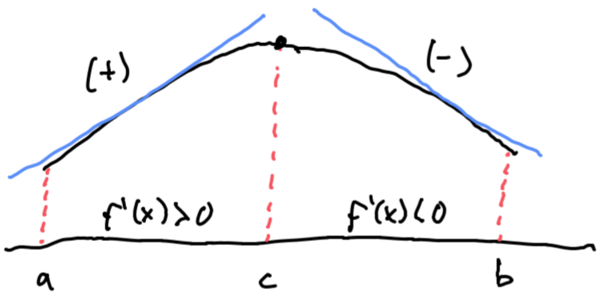


Fig. 15.8: Local Maximum

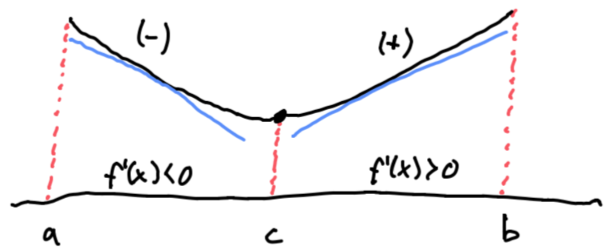


Fig. 15.9: Local Minimum

If the first derivative does not change sign then the derivative does not have a relative maximum or minimum value. See the figures below for why this is true.

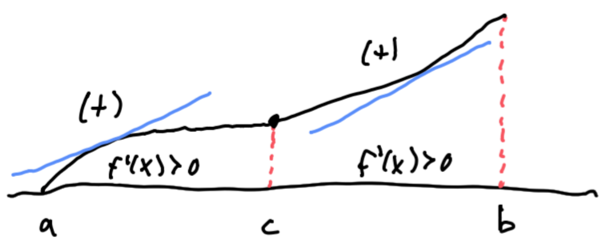


Fig. 15.10: $f'(x)$ is always positive.

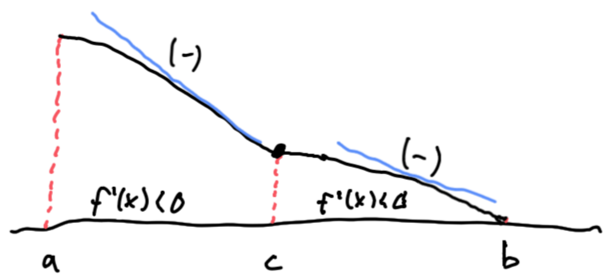


Fig. 15.11: $f'(x)$ is always negative.

It is highly recommended to check your work via graphing. If you find yourself confused about the material you should be graphing the function, first derivative, and second derivative for each of these examples. Verify that your calculations match what you see in the graph. Many students completely mess up their calculations with the derivatives and never bother to check what the graph actually looks like! Remember that you still need to show your work on an exam or homework assignment. This means graphing is only used to verify your results.

Example 15.3: Applying First Derivative Test

Find local extrema of $g(x) = (x^2 - 4)^{\frac{2}{3}}$

Solution. First we find the derivative

$$g'(x) = \frac{2}{3}(x^2 - 4)^{-\frac{1}{3}}(2x) = \frac{4x}{3(x^2-4)^{\frac{1}{3}}}$$

We see that $g'(x) = 0$ when $x = 0$ and $g'(x)$ does not exist when $x = \pm 2$.

So the critical numbers are located at $x = -2, 0, 2$. Now we can set up our table.

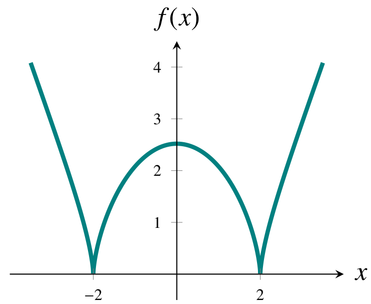


Fig. 15.12: Graph of $g(x)$

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
x	-3	-1	1	3
$g'(x)$	$g'(-3) \approx -2.339$	$g'(-1) \approx 0.942$	$g'(1) \approx -0.942$	$g'(3) \approx 2.339$
Sign	$-$	$+$	$-$	$+$
Behavior	decreasing	increasing	decreasing	increasing

Applying the First Derivative Test we can conclude the following:

Critical Value	Behavior at Critical Value	Conclusion
$x = -2$	$g(x)$ is decreasing then increasing	local minimum
$x = 0$	$g(x)$ is increasing then decreasing	local maximum
$x = 2$	$g(x)$ is decreasing then increasing	local minimum

The graph of g is pictured in Figure 15.12 for your reference.

It is recommended that you use the function table feature in your graphing calculator to assist you in evaluating your derivatives and functions. This will eliminate error and hopefully speed up your work. You do not need to show your arithmetic, simply report your values. Even if your values themselves are incorrect, it is still important to see that you can make correct conclusions from your wrong results.

The Mean Value Theorem & Other Theorems

The Mean Value Theorem, Rolle's Theorem, and a few other important theorems are discussed in Section 4.6 (pg. 290). Please read over the section and skim through the proofs of the respective theorems. Since you have now seen several examples, the more theoretical explanation of why we can use these approaches will hopefully make more sense.

The first four theorems are the more important theorems in the section and are merely stated below. The last theorem simply restates what we already know about intervals of increase and decrease but provides a proof of why we know this to be true.

Theorem 15.5: Rolle's Theorem

Let f be continuous on a closed interval $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$. Then there is at least one point c in (a, b) such that $f'(c) = 0$.

Theorem 15.6: Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Theorem 15.7: Zero Derivative Implies Constant Function

If f is differentiable and $f'(x) = 0$ at all points of an interval I, then f is a constant function on I.

The following theorem will be important when we discuss how to "undo" differentiation.

Theorem 15.8: Functions with Equal Derivatives Differ by a Constant

If two functions f and g have the property that

$$f'(x) = g'(x)$$

for all x of an interval I, then

$$f(x) - g(x) = C$$

on I, where C is a constant. In other words, f and g differ only by a constant.

Lecture 16

Graphing Functions Using the Second Derivative

Sections
4.5.3-4.5.6

Concavity

Concavity of a function is when the graph curves upward or downward. We obtain this information from whether the *first* derivative is increasing or decreasing.

Definition 16.1: Concavity

Let f be differentiable on an open interval I. The graph of f is

(i) **concave upward** on I if f' is increasing on the interval

(ii) **concave down** on I if f' is decreasing on the interval.

It is easier to visualize concavity using the figures below.

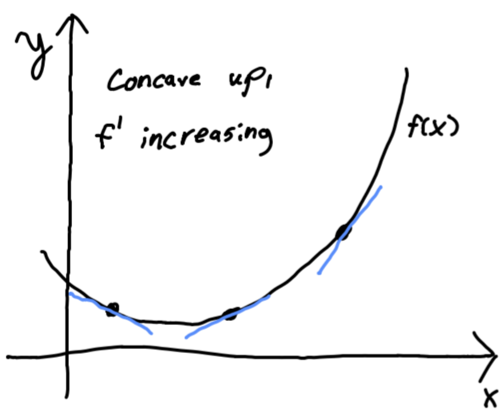


Fig. 16.1: Concave up function

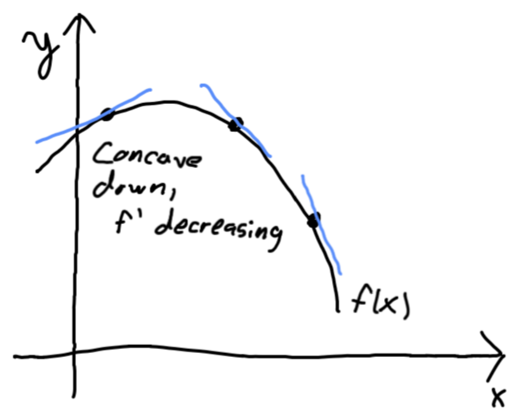


Fig. 16.2: Concave down function

In the figures above we can see that when f is concave up that the graph of f lies *above* its tangent lines. This means the graph is concave up when f' is increasing, in other words when $f''(x) > 0$. When f is concave down the graph of f lies *below* its tangent lines. This means that the graph is concave down when f' is decreasing, in other words when $f''(x) < 0$.

In summary, in order to determine concavity, we will need to apply the test for increasing and decreasing functions to the derivative of f, $f'(x)$. This means we'll need the derivative of the derivative, i.e. the *second* derivative of f, $f''(x)$. This test is summarized in the following theorem.

Theorem 16.1: Test for Concavity

Let f be a function whose second derivative exists on an open interval I.

(i) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(ii) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Note that common language for each of these cases is different in more advanced mathematics courses. Equivalent language is saying *convex* to mean concave upward, and *concave* to mean concave downward. You may have also seen these terms before in a geometry or trigonometry course.

Example 16.1: Applying Test for Concavity

Find the intervals for which $f(x) = \frac{x^3}{3} - 4x$ is concave up and concave down.

Solution. From Example 15.1 we know the first derivative. To use the test for concavity we need to find $f''(x)$ so we have

$$f'(x) = x^2 - 4 \quad \text{and} \quad f''(x) = 2x$$

Just as with the test for increasing and decreasing, we need intervals where f'' potentially changes sign. In other words, when $f''(x) = 0$. Clearly, $f''(x) = 0$ when $x = 0$. We use $x = 0$ to define the intervals we will test. We then set up our table below.

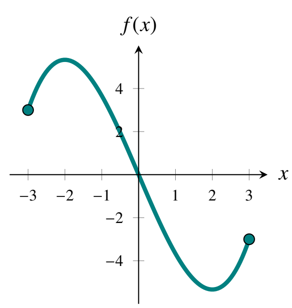


Fig. 16.3: Graph of $f(x)$

Interval	$(-\infty, 0)$	$(0, \infty)$
x	-1	1
$f''(x)$	$f''(-1) = -2$	$f''(1) = 2$
Sign	$-$	$+$
Behavior	concave down	concave up

So f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

⚠ When testing for concavity you must include any points for which the function itself does not exist. It can seriously mess up your results if you do not.

Points of Inflection

Definition 16.2: Point of Inflection

A **point of inflection** occurs when concavity of a function changes from concave up to concave down or from concave down to concave up.

To find *possible* points of inflection we use the following theorem.

Theorem 16.2: Points of Inflection

If $(c, f(c))$ is an inflection point then either $f''(c) = 0$ or f'' does not exist at $x = c$.

An important fact to note is that the converse (i.e. the reverse) of this statement is not true. In other words, it is possible that there is not an inflection point at $x = c$ when $f''(c) = 0$ or does not exist.

Example 16.2: Locating Points of Inflection

Locate the points of inflection for the following functions.

a.) $f(x) = \frac{x^3}{3} - 4x$

Solution. From example 16.1 we already have our table of second derivative information. We can see that at $x = 0$ the concavity changes from concave down to concave up. Thus, there is an inflection point at $x = 0$.

b.) $g(x) = x^4$

Solution. Finding the first and second derivative we have

$$g'(x) = 4x^3 \quad \text{and} \quad g''(x) = 12x^2$$

Now $g''(x) = 0$ when $x = 0$. We set up our table to test the intervals defined by this value below.

Interval	$(-\infty, 0)$	$(0, \infty)$
x	-1	1
$g''(x)$	12	12
Sign	+	+
Conclusion	concave up	concave up

In this case we see that concavity does not change. So even though $g''(x) = 0$ at $x = 0$ there is *not* an inflection point at $x = 0$.

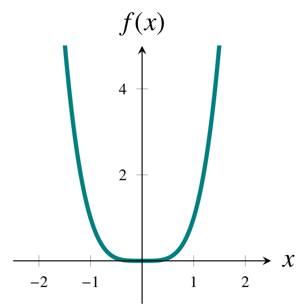


Fig. 16.4: Graph of $g(x)$

c.) $f(x) = \dfrac{x^2 + 1}{x^2 - 4}$

Solution. Finding the first and second derivative we have

$$f'(x) = -\frac{10x}{(x^2 - 4)^2} \quad \text{and} \quad f''(x) = \frac{10(3x^2 + 4)}{(x^2 - 4)^3}$$

In this case there are no points where $f''(x) = 0$. We do have that $f''(x)$ does not exist at $x = \pm 2$ (note that f also does not exist at these points). We then test the intervals defined by these values as seen in the table below.

Interval	$(-\infty,-2)$	$(-2,2)$	$(2,\infty)$
x	-3	0	3
$f''(x)$	$f''(-3)\approx 2.48$	$f''(0)\approx -0.625$	$f''(3)\approx 2.48$
Sign	$+$	$-$	$+$
Conclusion	concave up	concave down	concave up

We see that concavity changes at both $x = -2$ and at $x = 2$. Thus, there are points of inflection at $x = 2$ and $x = -2$.

To find points of inflection

(1) Find when $f''(c) = 0$ or $f''(c)$ does not exist.

(2) Test concavity on intervals defined by these points.

(3) If concavity changes then there is an inflection point at $x = c$.

The Second Derivative Test

In some cases we can also obtain maximum and minimum information using the second derivative. By examining Figures 16.1 and 16.2 we see that when a function is concave up or down, then the function had to change from decreasing to increasing (or vice versa) at some point. In other words, at a maximum or minimum value of the function. We know that maximum and minimum values must occur at a point c where $f'(c) = 0$.

What does this mean? Well, if a function is concave up or down at a critical point, then we have a minimum or maximum value at that point. This means that we can use the second derivative to actually determine whether a critical point is a local maximum or minimum value. This is known as the Second Derivative Test.

Theorem 16.3: Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

(i) If $f''(c) > 0$, then $f(c)$ is a **relative minimum** .

(ii) If $f''(c) < 0$, then $f(c)$ is a **relative maximum** .

(iii) If $f''(c) = 0$, the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you must use the first derivative test.

Example 16.3: Applying the Second Derivative Test

Use the second derivative test to find the relative extrema for $f(x) = \dfrac{x^3}{3} - 4x$.

Solution. From Example 15.2 we know that the critical numbers are $x = -2$ and $x = 2$. We also know that $f''(x) = 2x$.

Applying the Second Derivative Test we have the following information.

Critical Point	Value of $f''(x)$	Behavior	Conclusion
$x = -2$	$f''(-2) = -4$	$f''(x) < 0$	relative maximum
$x = 2$	$f''(2) = 4$	$f''(x) > 0$	relative minimum

It may seem easier to always use the second derivative test to determine whether you have a local maximum or minimum value at a certain point rather than bothering to determine where a function is increasing or decreasing. However, recall that in Theorem 16.3 part (iii) we have a case when the second derivative test can fail, i.e. when $f''(c) = 0$.

Example 16.4: When the Second Derivative Test Fails

Use the second derivative test to find the relative extrema for the function $h(x) = -3x^5 + 5x^3$

Solution. We need to find the first derivative, second derivative, and the critical points. We have

$$h'(x) = -15x^4 + 15x^2 = 15x^2(1 - x^2) \quad \text{and} \quad h''(x) = -60x^3 + 30x = 30(-2x^3 + x)$$

The critical points are $x = -1, 0, 1$. Applying the second derivative test we have

Critical Point	Value of $h''(x)$	Behavior	Conclusion
$x = -1$	$h''(-1) = 30$	$f''(x) > 0$	relative minimum
$x = 1$	$h''(1) = -30$	$f''(x) < 0$	relative maximum
$x = 0$	$h''(0) = 0$	$f''(x) = 0$	test fails

Since the test fails at $x = 0$ you must use the first derivative test to determine if there is a relative maximum or minimum located at $x = 0$.

You should also keep in mind if the function f itself (and not just the derivatives of f) is actually defined at $f(c)$. Just as in algebra class (and in the rest of mathematics), it is important to always take into account the domain and range of a function. It makes no sense to determine information about points where a function is not even defined!

Students constantly confuse the first and second derivative tests with the tests for increasing/decreasing and the test for concavity. Applying the first and second derivative tests is the point when you are determining if a critical point is a maximum or minimum value. This involves using information from tests for increasing/decreasing or concavity. Be sure you understand the difference!

Lecture 17

Limits at Infinity

Section
4.6; 4.8.3

Using Infinity

Before we can talk about limits involving infinity you must understand some key things about infinity. First, it is important to remember that **infinity is not a number** itself. It represents a concept. If you go on in mathematics you will learn that there are even different types of infinities!

Because of this, we are not always permitted to carry out the usual operations between infinities. How much is $\infty - \infty$? It turns out this isn't even a meaningful question! The expression $\infty - \infty$ is what is known as an **indeterminate form**. We will learn how to deal with limits involving these in a later lesson.

Definition 17.1: Indeterminate Forms

$$1^\infty \qquad 0^0 \qquad \infty^0 \qquad 0 \cdot \infty \qquad \frac{0}{0} \qquad \frac{\infty}{\infty} \qquad \infty - \infty$$

Some of these indeterminate forms may seem surprising. We're used to the concepts that a number minus itself is 0 and a number divided by itself is 1. But **infinity if not a number**. Similarly, 0^0 and 1^∞ do not mean what you expect. In particular, $1^\infty \neq 1 \cdot 1 \cdots$ but rather it is defined to be $\lim_{x\to\infty} f(x)^{g(x)}$ where $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to\infty} g(x) = \infty$. Given different choices for f and g, we can make this limit go to 1, e, ∞, or any number of other options. 0^0 is defined similarly, though we note that *as a number*, 0^0 does equal 1. More detailed explanations for why these are indeterminate forms are outside of our current scope. The oversimplified explanation is that the symbol ∞ is **always** an abbreviation for a concept involving limits, so many operations using it aren't defined.

What are the "legal" operations that can be carried out in regards to infinity? The following are common cases you may see as you evaluate limits. Note that these are not actual operations, merely guidelines for results you may see. These are presented without proof because some involve higher level mathematics knowledge than we see in this course. These should merely be used as a reference as to what is acceptable and what is not when dealing with infinity.

Definition 17.2: Properties of Infinity

Let k be a real number.

$$\infty \pm k = \infty \qquad \infty \cdot \infty = \infty \qquad 0^\infty = 0$$

$$\infty + \infty = \infty \qquad (\pm k)\infty = \pm\infty, \text{ for } k \neq 0 \qquad \infty^\infty = \infty$$

$$0^k = \begin{cases} 0, & \text{if } k > 0 \\ \infty, & \text{if } k < 0 \end{cases} \qquad k^\infty = \begin{cases} \infty, & \text{if } k > 1 \\ 0, & \text{if } 0 < k < 1 \end{cases}$$

The following is an important concept when dealing with real numbers. For positive real numbers, we see that

$$\frac{1}{\text{Small Number}} = \text{Large Number} \qquad \text{and} \qquad \frac{1}{\text{Large Number}} = \text{Small Number}$$

Mathematically, these cases can be represented as follows. (Again, note that these are not actual operations, merely a way of representing the concept of what is happening in each case.)

Definition 17.3: Properties of Division Involving Infinity

Let k be a real number.

$$\frac{0}{k} = 0, \quad \frac{0}{\infty} = 0, \quad \frac{k}{\infty} = 0, \quad \frac{k}{0} = \pm\infty, \quad \frac{\infty}{k} = \infty, \quad \frac{\infty}{0} = \infty$$

Remark: When actually writing out your work as you evaluate a limit it is never acceptable to write

$$\lim_{x \to a} f(x) = \frac{1}{\infty}$$

since this is not a correct mathematical statement. However, often you will find yourself writing this very term to keep track of your work. The acceptable way to write this would be

$$\lim_{x \to a} f(x) = \frac{1}{\infty} = 0$$

This way you are properly communicating that you understand this value to be 0 rather than a nonsensical mathematical term. It may seem tedious but notation is important in mathematics so get used to it!

Limits at Infinity

Consider the function $f(x) = \dfrac{1}{x^2}$

Now suppose that we let x get larger and larger. In other words let x approach infinity. In Table 17.1 we see some values for $f(x)$ as $x \to \infty$.

x goes to positive infinity

x	1	10	100	1000	100000	...	∞
$f(x)$	1	0.1	0.001	0.0001	0.00001	...	0

$f(x)$ approaches 0

Table 17.1: Values of $f(x) = \frac{1}{x^2}$ as $x \to \infty$

In this case we see that our function value will get smaller and smaller, so $f(x)$ gets closer and closer to zero. Letting x approach positive or negative infinity is what is known as a **limit at infinity**.

Definition 17.4: Limits at Infinity

If $f(x)$ becomes arbitrarily close to a finite number L

- For all x sufficiently large and positive, then

$$\lim_{x\to\infty} f(x) = L$$

and we say that the limit of $f(x)$ as x goes to infinity is L.

- For all x sufficiently large *in magnitude*, then

$$\lim_{x\to-\infty} f(x) = L$$

and we say that the limit of $f(x)$ as x goes to negative infinity is L.

Horizontal Asymptotes

Similar to infinite limits, limits at infinity allow us to find **horizontal asymptotes** of functions.

Definition 17.5: Horizontal Asymptotes

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to-\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to\infty} f(x) = L$$

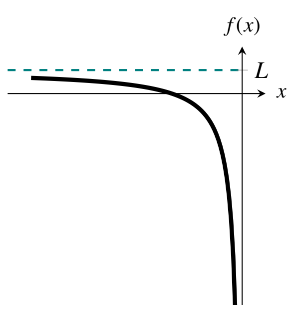


Fig. 17.1: $\lim_{x\to-\infty} f(x) = L$

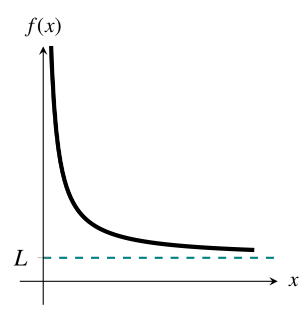


Fig. 17.2: $\lim_{x\to\infty} f(x) = L$

Slant Asymptotes

In addition to horizontal and vertical asymptotes, it is possible for a function to have a **slant asymptote** (also referred to as oblique asymptotes). To find a slant asymptote you must carry out polynomial long division for $f(x)$. In other words, divide the numerator by the denominator. If you need to review polynomial long division consult page 142 of the *Just in Time* text.

Evaluating Limits at Infinity

Similar strategies for finding finite limits are used here. All of the same limit laws apply. The behavior of the function $f(x) = \frac{1}{x^2}$ leads us to an important theorem in regards to limits at infinity.

Theorem 17.1

If $n > 0$ is a rational number then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0$$

If $n > 0$ is a rational number such that x^n is defined for all x then

$$\lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Example 17.1: Limits at Infinity and Limit Laws

Evaluate $\lim_{x\to\infty}\left(5 + \frac{1}{x}\right)$

Solution.

$$\begin{aligned} \lim_{x\to\infty}\left(5 + \frac{1}{x}\right) &= \lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{1}{x} && \text{Sum of limits} \\ &= 5 + 0 = 5 && \text{Limit of a constant and Theorem 16.1} \end{aligned}$$

The following example demonstrates limits of polynomials at infinity.

Example 17.2: Limits of Polynomials at Infinity

Evaluate $\lim_{x\to\infty}\left(2x^4 - x^2 - 8x\right)$

Solution. We are not able to use direct substitution here. If we were to just plug in ∞ we would obtain

$$\lim_{x\to\infty}\left(2x^4 - x^2 - 8x\right) = 2(\infty)^4 - (\infty)^2 - 8(\infty)$$

We have seen previously that this yields and indeterminate form. So this is not a mathematically correct statement. In order to better see this limit we first factor out the highest degree term which yields

$$\begin{aligned} \lim_{x\to\infty}\left(2x^4 - x^2 - 8x\right) &= \lim_{x\to\infty}\left(x^4\left(2 - \frac{1}{x^2} - \frac{8}{x^3}\right)\right) && \text{Factor out } x^4 \\ &= \left(\lim_{x\to\infty} x^4\right)\left(\lim_{x\to\infty}\left(2 - \frac{1}{x^2} - 8\frac{1}{x^3}\right)\right) && \text{Apply limit laws} \\ &= \left(\lim_{x\to\infty} x^4\right)\left(\lim_{x\to\infty} 2 - \lim_{x\to\infty}\frac{1}{x^2} - 8\lim_{x\to\infty}\frac{1}{x^3}\right) \\ &= (\infty)^4(2 - (0) - 8(0)) = \infty && \text{By Theorem 16.1} \end{aligned}$$

By using the properties given at the beginning of the lesson we see that this limit is positive infinity.

This example leads to another important theorem

Theorem 17.2: Limits of Polynomials at Infinity

Let n be a positive integer and let $p(x)$ be an nth degree polynomial with leading term $a_n x^n$.

a.) $\lim_{x\to\infty} x^n = \infty$

b.) $\lim_{x\to-\infty} x^n = \infty$, when n is even

c.) $\lim_{x\to-\infty} x^n = -\infty$, when n is odd

d.) $\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_n x^n = \pm\infty$. Where the sign depends on the degree of the polynomial and the sign of the leading coefficient.

When dealing with rational functions as $x \to \pm\infty$, the behavior of the numerator with respect to the denominator is not always clear initially. In order to analyze these limits, we ideally want them to have the form of functions for which we know the limit as $x \to \pm\infty$, namely $\frac{1}{x^n}$. We can achieve this form by dividing both the numerator and denominator by the highest power term in the denominator.

Example 17.3: Limit at Infinity of a Rational Function

a.) Evaluate $\lim_{x\to\infty} \dfrac{11x+2}{2x^3-1}$

Solution.

$$\begin{aligned}
\lim_{x\to\infty} \frac{11x+2}{2x^3-1} &= \lim_{x\to\infty} \frac{\frac{11x}{x^3}+\frac{2}{x^3}}{\frac{2x^3}{x^3}-\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{11}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}} && \text{Divide by } x^3 \text{ and simplify} \\
&= \frac{11\lim_{x\to\infty}\frac{1}{x^2}+2\lim_{x\to\infty}\frac{1}{x^3}}{\lim_{x\to\infty} 2-\lim_{x\to\infty}\frac{1}{x^3}} && \text{Apply limit laws} \\
&= \frac{11(0)+2(0)}{2-(0)} = \frac{0}{2} = 0 && \text{By Theorem 16.1}
\end{aligned}$$

b.) Evaluate $\lim_{x\to\infty} \dfrac{5x^2+8x-3}{3x^2+2}$

Solution.

$$\begin{aligned}
\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} &= \lim_{x\to\infty} \frac{\frac{5x^2}{x^2}+\frac{8x}{x^2}-\frac{3}{x^2}}{\frac{3x^2}{x^2}+\frac{2}{x^2}} = \lim_{x\to\infty} \frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}} && \text{Divide by } x^2\text{, simplify} \\
&= \frac{\lim_{x\to\infty} 5+8\lim_{x\to\infty}\frac{1}{x}-3\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 3+2\lim_{x\to\infty}\frac{1}{x^2}} && \text{Applying limit laws} \\
&= \frac{5+8(0)+3(0)}{3+2(0)} = \frac{5}{3} && \text{By Theorem 16.1}
\end{aligned}$$

c.) Evaluate $\lim_{x\to\infty} \dfrac{2x^2-3}{7x+4}$

Solution.

$$\lim_{x\to\infty} \frac{2x^2-3}{7x+4} = \lim_{x\to\infty} \frac{\frac{2x^2}{x}-\frac{3}{x}}{\frac{7x}{x}+\frac{4}{x}} = \lim_{x\to\infty} \frac{2x-\frac{3}{x}}{7+\frac{4}{x}} \qquad \text{Divide by } x \text{ and simplify}$$

$$= \frac{2\lim_{x\to\infty} x - 3\lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty} 7 + 4\lim_{x\to\infty}\frac{1}{x}} \qquad \text{Apply limit laws}$$

$$= \frac{2(\infty)-3(0)}{7+4(0)} \qquad \text{By Theorem 16.1}$$

$$= \frac{\infty}{7} = \infty \qquad \text{By properties of infinity}$$

Each of the limits in the previous example demonstrates the behavior seen in algebra courses for finding horizontal asymptotes.

Limits at infinity that contain roots can be a little trickier to deal with than the examples we have seen thus far. Note that $\sqrt{x^2}$ is equivalent to $|x|$, the absolute value function. In other words,

$$\sqrt{x^2} = |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

This means that we must pay attention to whether we are approaching positive or negative infinity.

Example 17.4: Limit at Infinity of a Root Function

Evaluate the following limits for $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$

a.) $\lim_{x\to\infty} f(x)$

Solution. In this case, dividing by the highest power term in the denominator means that we're actually dividing by x^2 when under the square root in the numerator. It may make more sense to factor and cancel terms rather than divide in problems like this.

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)} = \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)}$$

Since $x\to\infty$ we use the fact that $\sqrt{x^2} = x$ for $x > 0$. Thus,

$$\lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)} = \lim_{x\to\infty} \frac{\cancel{x}\sqrt{2+\frac{1}{x^2}}}{\cancel{x}\left(3-\frac{5}{x}\right)} = \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} \qquad \text{Cancel common terms}$$

$$= \frac{\sqrt{\lim_{x\to\infty} 2 + \lim_{x\to\infty} \frac{1}{x^2}}}{\lim_{x\to\infty} 3 - 5\lim_{x\to\infty} \frac{1}{x}} \qquad \text{Apply limit laws}$$

$$= \frac{\sqrt{2+0}}{3-5(0)} = \frac{\sqrt{2}}{3} \qquad \text{Theorem 16.1}$$

b.) $\lim_{x\to-\infty} f(x)$

Solution. Similar to the last example we get to the point

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)} = \lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)}$$

Since we have $x \to -\infty$ we use the fact that $\sqrt{x^2} = -x$ for $x < 0$. Thus,

$$\lim_{x\to-\infty} \frac{-x\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)} = \lim_{x\to-\infty} \frac{-\cancel{x}\sqrt{2+\frac{1}{x^2}}}{\cancel{x}\left(3-\frac{5}{x}\right)} = \lim_{x\to-\infty} \frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} \qquad \text{Cancel common terms}$$

$$= \frac{-\sqrt{\lim_{x\to\infty} 2 + \lim_{x\to-\infty} \frac{1}{x^2}}}{\lim_{x\to-\infty} 3 - 5\lim_{x\to-\infty} \frac{1}{x}} \qquad \text{Apply limit laws}$$

$$= -\frac{\sqrt{2+0}}{3-5(0)} = -\frac{\sqrt{2}}{3} \qquad \text{Theorem 16.1}$$

The following theorem is listed in your textbook. Note that it merely summarizes information that we already know from algebra. Now we are at the level that when we determining asymptotes of a function we should be making precise statements involving our understanding of *limits*.

Theorem 17.3: Asymptotes of Rational Functions

Let $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ has leading term $a_n x^n$ and $q(x)$ has leading term $b_m x^m$ where $a_n \neq 0$, $b_m \neq 0$.

- **Horizontal Asymptotes:**
  - If $m < n$ then $\lim\limits_{x \to \pm\infty} f(x) = 0$ and the line $y = 0$ is a horizontal asymptote of $f(x)$.
  - If $m = n$ then $\lim\limits_{x \to \pm\infty} f(x) = \dfrac{a_n}{b_n}$ and the line $y = \frac{a_n}{b_n}$ is a horizontal asymptote of $f(x)$.
  - If $m > n$ then $\lim\limits_{x \to \pm\infty} f(x) = \infty$ or $-\infty$ and f has no horizontal asymptotes.
- **Slant Asymptotes:** If $m = n + 1$ then $\lim\limits_{x \to \pm\infty} f(x) = \infty$ or $-\infty$. Note that a slant asymptote will be the equation of a line.
- **Vertical Asymptotes:** The vertical asymptotes of $f(x)$ occur when $\lim_{x \to a^\pm} f(x) = \pm\infty$.

Remark: Students often mistakenly use this theorem in regards to limits involving root functions. This is not an appropriate use of this theorem! The first statement of this theorem indicates that this applies to rational functions only!

Lecture 18

Optimization

Section
4.7

Introduction

In previous lectures we learned how to determine the maximum and minimum values of a function. In this lecture we will now learn how this knowledge can be used in real world applications.

In the real world we often want to find a way minimize or maximize a process. This is a process known as *optimization*. We'll demonstrate how to do this with the derivative through a few of the classic optimization problems in calculus. As in the section on related rates it is recommend that you use the GASCAP method of problem solving or a similar organizational method for your work.

Maximizing Area

Example 18.1: Maximizing Area

A rancher has 400 feet of fencing with which to enclose two adjacent rectangular corrals. What dimensions should be used so that the enclosed area will be a maximum? Also find the area of each pen.

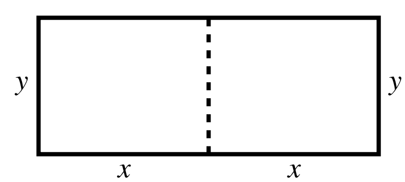
Solution.

Given: 400 ft of fencing and 2 rectangular pens. The equation for the perimeter is

$$4x + 2y = 400\,\text{ft}$$

Asked: We need to find the maximum area of the two pens. The combined area will be given by $A = 2xy$

Sketch:



Compute: The area formula is a function of two variables x and y. Solving our perimeter formula for one of the variables allows us to simpify our area equation. It doesn't really matter which variable we solve for but in general it is good practice to choose the one that will simplify equations in which it is used. This is not always clear initially but becomes easier with more practice. Here we solve for y and obtain

$$400 - 4x = 2y \implies y = 200 - 2x$$

Plugging this into our equation for A we have

$$A(x) = 2x(200 - 2x) = 400x - 4x^2$$

Now to find the maximum area we need to find the maximum value of the function A. We know from Lesson 8 that extreme values exist only at critical points so we first need to find values of x for which $A'(x) = 0$ where $A'(x) = 400 - 8x$

$$400 - 8x = 0 \quad \Longrightarrow \quad x = 50$$

We need to confirm that this is a maximum value using either the first or second derivative test. Sometimes you will need to use one test over the other but here it does not matter which you use so we demonstrate both.

Using the first derivative test:

Interval	$(-\infty, 50)$	$(50, \infty)$
x	40	60
$f'(x)$	$f'(40) = 80$	$f'(60) = -80$
Sign	$+$	$-$
Behavior	increasing	decreasing

Since $f'(x)$ changes from increasing to decreasing at $x = 50$ by the first derivative test we have that $x = 50$ is a maximum value of A.

Using the second derivative test:
Finding the second derivative we have $f''(x) = -8$. This is a constant function so we will always have $f''(x) < 0$. By the second derivative test we have that $x = 50$ is a maximum. We see that in this case it is much easier to apply the second derivative test. This saves us from needing to determine whether the function is increasing or decreasing when we apply the first derivative test.

Now that we have confirmed that $x = 50$ is a maximum we can find the value of our maximum area as well as our missing dimension y. Using our equation for y we have

$$y = 200 - 2x = 200 - 2(50) = 100$$

and so our maximum area is

$$A = 2x \cdot y = 2(50)(100) = 10000$$

Answer: The dimensions of each pen is $x = 50$ and $y = 100$ where the maximum area of each pen is

$$\frac{1}{2}A = \frac{1}{2}(10000) = 5000\,\text{ft}^2$$

In the previous example the amount of fencing available placed a restriction on the area of the pen that could be created. Values that have this effect are referred to as **constraints**. The equation that produces this value is often referred to as the **constraint equation**.

Maximizing Volume

Example 18.2: Maximizing Volume

A manufacturer wants to design an open box with a square base and a surface area of 108 square inches. What dimensions produce a box with maximum volume?

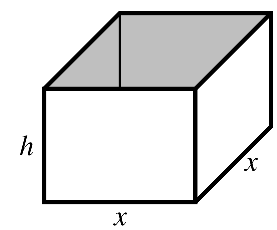
Solution.

Given: We are given that the surface area is 108 in^2. The surface area is given by the equation

$$x^2 + 4xh = 108$$

Where x^2 accounts for the square base and the $4xh$ accounts for the area of the four sides.

Sketch:



Asked: Find x and h that will give the maximum volume which is given by the equation $V = x^2 \cdot h$.

Compute: To eliminate one of the variables in V we solve the constraint equation for h:

$$x^2 + 4xh = 108 \implies h = \frac{108 - x^2}{4x}$$

Substituting this into our volume equation we obtain

$$V(x) = x^2\left(\frac{108 - x^2}{4x}\right) = 27x - \frac{x^3}{4}$$

In order to find the maximum value of the volume we need to first find the critical points. Our derivative is $V'(x) = 27 - \frac{3}{4}x^2$. Setting $V' = 0$ and solving for x we have

$$27 - \frac{3x^2}{4} = 0 \qquad \implies \qquad x = \pm 6$$

A negative length does not make sense in this context so we only consider the critical point at $x = 6$. We use the second derivative test here where $V''(x) = \frac{-6}{4}x$. Since

$$V''(6) = -\frac{3}{2}(6) = -3(3) = -9 < 0$$

by the second derivative then the critical point at $x = 6$ is a maximum. Thus, one of our dimensions is $x = 6$. To find h we plug in $x = 6$ to our equation for h and find

$$h = \frac{108 - (6)^2}{4(6)} = 3$$

Answer: The dimensions of the box are $x = 6$ and $h = 3$ with a maximum volume of

$$V = (6)^2 \cdot 3 = 108\text{ in}^3$$

Minimum Distance

Example 18.3: Find Minimum Distance

Determine which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$.

Solution.
Given: We are given the point $(0, 2)$ and the graph $y = 4 - x^2$.

Asked: Find the minimum distance between the point and the graph of y. Recall that the distance between two points is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In this case, we want the distance between $(0, 2)$ and an unknown point (x, y). Since (x, y) is on the graph of y and $y = 4 - x^2$ our second point is actually $(x, 4 - x^2)$. So,

$$d = \sqrt{(x-0)^2 + (y-2)^2} = \sqrt{(x-0)^2 + (4 - x^2 - 2)^2} = \sqrt{x^4 - 3x^2 + 4}$$

and so the function we aim to minimize is

$$d(x) = \sqrt{x^4 - 3x^2 + 4}$$

Compute: You can find $d'(x)$ directly and then use the usual tests but to simplify things we will actually just consider the fact that $d(x)$ will be smallest when the quantity under the square root is smallest. So instead we will actually minimize $f(x) = x^4 - 3x^2 + 4$. (An explanation why we are doing this is given after the example).

Following the same logic in the first two examples we find the first derivative, locate all critical points, and then apply an appropriate derivative test. We have $f'(x) = 4x^3 - 6x$. Setting this equal to zero we have

$$4x^3 - 6x = 0$$
$$2x(2x^2 - 3) = 0$$

The critical points of f are at $x = 0$ and $x = \pm\frac{3}{2}$. Our second derivative is $f''(x) = 12x^2 - 6$. Evaluating the second derivative at each of these values we have

Critical Point	Value of $f''(x)$	Behavior	Conclusion
$x = -\sqrt{\frac{3}{2}}$	$f''\left(-\sqrt{\frac{3}{2}}\right) = \frac{\sqrt{12}}{7}$	$f''(x) > 0$	relative minimum
$x = 0$	$f''(0) = -\frac{3}{2}$	$f''(x) < 0$	relative maximum
$x = \sqrt{\frac{3}{2}}$	$f''\left(\sqrt{\frac{3}{2}}\right) = \frac{\sqrt{12}}{7}$	$f''(x) > 0$	relative minimum

So by the second derivative test the critical point at $x = 0$ is a relative maximum and $\pm\sqrt{\frac{3}{2}}$ are relative minimums. So, $f(x)$ is smallest at $x = -\sqrt{\frac{3}{2}}$ and $x = \sqrt{\frac{3}{2}}$ and so $d(x)$ will also be smallest at $x = -\sqrt{\frac{3}{2}}$ and $x = \sqrt{\frac{3}{2}}$. To identify these points we evaluate y at these x values and obtain

$$y = 4 - \left(-\sqrt{\frac{3}{2}}\right)^2 = \frac{5}{2} \quad \text{and} \quad y = 4 - \left(\sqrt{\frac{3}{2}}\right)^2 = \frac{5}{2}$$

Answer: The points on $y = 4 - x^2$ closest to the point $(0, 2)$ are

$$\left(-\sqrt{\frac{3}{2}}, \frac{5}{2}\right) \quad \text{and} \quad \left(\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$$

Now why did we choose to look at the quantity underneath the square root instead of the whole function in the last example? Finding the derivative is the easy part, the challenge then comes with locating critical points. The derivative is

$$d'(x) = \frac{4x^3 - 6x}{2\sqrt{x^4 - 3x^2 + 4}}$$

Locating critical points requires us to find points where $d'(x) = 0$. Since this is a quotient of two functions, $d'(x)$ is zero when the numerator is zero and so we need to look at when $4x^3 - 6x = 0$. As in the example above, we need to factor here. If we were to solve for x using typical algebraic steps we would need to divide each side by x. We run into a problem here since in this case x could be zero. This is why paying attention to the domain of a function is important! As above we locate the same 3 critical points $x = 0$ and $x = \pm\frac{3}{2}$.

We must also consider when $d'(x)$ does not exist. This will occur when the denominator is zero and since we also have a square root in the denominator we must also look at when $x^4 - 3x^2 + 4$ is less than zero. By examining a graph of this function you will see that this function is never zero and always positive. So there are no points where $d'(x)$ does not exist.

The main point here is to make sure you are looking at the functions you are maximizing very carefully. Full use of your algebra knowledge will be required for these problems. For these problems the "possible" step of GASCAP is maybe the most important.

Lecture 19

L'Hôpital's Rule

Section
4.8

Indeterminate Forms and L'Hôpital's Rule

In previous lectures we saw infinite limits and limits as x goes to infinity. Many of these limits can be difficult to handle using techniques we have learned thus far. Recall that many of these limits can have an **indeterminate form**. As a reminder the types of indeterminate forms are given below.

Definition 19.1: Indeterminate Forms

$$1^\infty \qquad 0^0 \qquad \infty^0 \qquad 0\cdot\infty \qquad \frac{0}{0} \qquad \frac{\infty}{\infty} \qquad \infty-\infty$$

Now that we know how to find the derivative of a function we can make use of the following rule.

Theorem 19.1: L'Hôpital's Rule

Suppose f and g are differentiable functions on an open interval (a,b) containing c and that $g'(x) \neq 0$ for all x in (a,b). If $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$, $\frac{\infty}{\infty}$ then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

⚠ L'Hôpital's Rule can only be used for these two types of indeterminate forms! If a limit does not have form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then it must first be rewritten or simplified (if possible) to give it the necessary form *before* you can apply L'Hôpital's Rule.

Indeterminate Forms $0/0$ and ∞/∞

We start by examining how to apply L'Hôpital's Rule for limits that already have the basic indeterminate forms required by the theorem namely, $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

It is important not to forget the differentiation rules you have just learned. The most common mistake students make when applying L'Hôpital's Rule is calculating derivatives incorrectly.

Since the derivative is now just a part of a calculation, it may be worth doing derivative calculations off to the side so you don't confuse your work. This will also give you additional practice in taking derivatives. Note that the derivative calculations themselves will be skipped in the examples presented in this lesson.

Example 19.1: Indeterminate form $\frac{0}{0}$

Evaluate $\lim_{x\to 3} \dfrac{x^2-2x-3}{x-3}$

Solution. Using direct substitution we see that

$$\lim_{x\to 3} \frac{x^2-2x-3}{x-3} = \frac{3^3-2(3)-3}{3-3} = \frac{0}{0}$$

Since this has the indeterminate form $\frac{0}{0}$, we can apply L'Hôpital's Rule:

$$\begin{aligned} \lim_{x\to 3} \frac{x^2-2x-3}{x-3} &\overset{\text{H}}{=} \lim_{x\to 3} \frac{\frac{d}{dx}[x^2-2x-3]}{\frac{d}{dx}[x-3]} && \text{Apply L'Hôpital's Rule} \\ &= \lim_{x\to 3} \frac{2x-2}{1} \\ &= \frac{2(3)-2}{1} = 4 && \text{By direct substitution} \end{aligned}$$

The symbol $\overset{\text{H}}{=}$ is used to indicate that L'Hôpital's Rule has been applied. It is "polite" in mathematics to indicate when special rules or techniques are applied so that others can follow what you're doing.

Example 19.2: Indeterminate Form $\frac{\infty}{\infty}$

Evaluate the following limits.

(a) $\lim_{x\to\infty} \dfrac{x^3}{x+2}$

Solution. Using direct substitution we have

$$\lim_{x\to\infty} \frac{x^3}{x+2} = \frac{\infty^3}{\infty+2} = \frac{\infty}{\infty}$$

We have the form $\frac{\infty}{\infty}$ so we can apply L'Hôpital's Rule

$$\begin{aligned} \lim_{x\to\infty} \frac{x^3}{x+2} &\overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}[x^3]}{\frac{d}{dx}[x+2]} && \text{Apply L'Hôpital's Rule} \\ &= \lim_{x\to\infty} \frac{3x^2}{1} \\ &= 3(\infty)^2 = \infty && \text{By direct substitution} \end{aligned}$$

(b) $\lim_{x\to\infty} \frac{\ln(x)}{x}$

Solution. Since this has the indeterminate form $\frac{\infty}{\infty}$, we can apply L'Hôpital's Rule.

$$\lim_{x\to\infty} \frac{\ln(x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[x]} \qquad \text{Apply L'Hôpital's Rule}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1}$$

$$= \lim_{x\to\infty} \frac{1}{x} \qquad \text{Simplifying}$$

$$= \frac{1}{\infty} = 0 \qquad \text{By direct substitution}$$

It is possible to apply L'Hôpital's Rule more than once. Each time you apply the rule you need to make sure that the resulting expression still has the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

We must also take care that at each step we have $g'(x) \neq 0$. In other words, you should stop when you have a constant in the denominator.

Example 19.3: Applying L'Hôpital's Rule multiple times

Evaluate $\lim_{x\to-\infty} \frac{x^2}{e^{-x}}$

Solution. Using direct substitution we see that

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} = \frac{(-\infty)^2}{e^{-(-\infty)}} = \frac{(-\infty)^2}{e^{\infty}} = \frac{\infty}{\infty}$$

Since we have the form $\frac{\infty}{\infty}$, we can apply L'Hôpital's Rule

$$\lim_{x\to-\infty} \frac{x^2}{e^{-x}} \overset{\text{H}}{=} \lim_{x\to-\infty} \frac{\frac{d}{dx}[x^2]}{\frac{d}{dx}[e^{-x}]} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}}$$

This still has the form $\frac{\infty}{\infty}$ so we apply L'Hôpital's Rule a second time.

$$\lim_{x\to-\infty} \frac{2x}{-e^{-x}} \overset{\text{H}}{=} \lim_{x\to-\infty} \frac{\frac{d}{dx}[2x]}{-\frac{d}{dx}[e^{-x}]} \qquad \text{Apply L'Hôpital's Rule}$$

$$= \lim_{x\to-\infty} \frac{2}{e^{-x}}$$

$$= \frac{2}{e^{-(-\infty)}} = \frac{2}{e^{\infty}} = \frac{1}{\infty} = 0 \qquad \text{By direct substitution}$$

Indeterminate Form $0 \cdot \infty$

We now proceed with some examples of limits which are *not* initially in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The goal here is to manipulate the expression in the limit so that it has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Before we continue, you should recall the fact that the product of two functions can be written as

$$fg = \frac{f}{\frac{1}{g}} = \frac{g}{\frac{1}{f}}$$

This trick will be very useful when re-writing limits in the form necessary to apply L'Hôpital's Rule.

Example 19.4: Indeterminate form $0 \cdot \infty$

Evaluate $\lim_{x\to 0^+} x\ln(x)$

Solution. By applying limit laws we have

$$\lim_{x\to 0^+} x \cdot \lim_{x\to 0^+} \ln(x)$$

Recall that $\lim_{x\to 0^+} \ln(x) = -\infty$. So this limit has the indeterminate form $0 \cdot \infty$. This is not one of the forms necessary to apply L'Hôpital's rule. We must first re-write this product using the form shown above to obtain

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}}$$

Now the limit has the indeterminate form $\frac{\infty}{\infty}$ so we can now apply L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} &\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[\frac{1}{x}]} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} && \text{Apply L'Hôpital's Rule} \\
&= \lim_{x\to 0^+} \frac{-x^2}{x} && \text{Rewriting} \\
&= \lim_{x\to 0^+} -x && \text{Simplifying} \\
&= 0 && \text{By direct substitution}
\end{aligned}$$

Indeterminate Form $\infty - \infty$

The indeterminate form $\infty - \infty$ can be handled using several different strategies. The main goal is to create a single quotient. We can do this by finding a common denominator, rationalizing, or factoring out common terms.

Example 19.5: Indeterminate Form $\infty - \infty$

Evaluate $\lim\limits_{x \to 0} \left(\dfrac{1}{\sin(x)} - \dfrac{1}{x} \right)$

Solution. This limit has the indeterminate form $\infty - \infty$. This is not one of the forms necessary to apply L'Hôpital's rule. We must first re-write the limit. In this case, by finding a common denominator.

$$\lim_{x \to 0} \frac{x - \sin(x)}{x \sin(x)}$$

The limit now has form $\frac{0}{0}$ and so we can apply L'Hôpital's rule. Evaluating this limit will involve two applications of L'Hôpital's rule. Take care when taking derivatives. Note that the derivatives in the denominator will involve the product rule.

$$\begin{aligned}
\lim_{x \to 0} \frac{x - \sin(x)}{x \sin(x)} &\overset{\text{H}}{=} \lim_{x \to 0} \frac{\frac{d}{dx}[x - \sin(x)]}{\frac{d}{dx}[x \sin(x)]} && \text{Apply L'Hôpital's rule} \\
&= \lim_{x \to 0} \frac{1 - \cos(x)}{\sin(x) + x \cos(x)} && \text{Has indeterminate form } \frac{0}{0} \\
&\overset{\text{H}}{=} \lim_{x \to 0} \frac{\frac{d}{dx}[1 - \cos(x)]}{\frac{d}{dx}[\sin(x) + x \cos(x)]} && \text{Apply L'Hôpital's again} \\
&= \lim_{x \to 0} \frac{\sin(x)}{2\cos(x) - x \sin(x)} \\
&= \frac{\sin(0)}{2\cos(0) - (0)\sin(0)} = \frac{0}{2} = 0 && \text{By direct substitution}
\end{aligned}$$

It may not always be immediately evident which strategy you should use. Through practice, you will be able to better recognize when each strategy will work best for a particular limit. Until you have developed this intuition, try out each strategy to see which works best.

Indeterminate Forms 1^∞, 0^∞, and 0^0

There are a few different ways to think about limits that involve the indeterminate forms 1^∞, 0^∞, and 0^0. The method presented in these notes differs from the one in the text but may be easier to remember since it is similar to how we carried out logarithmic differentiation.

In general, limits with these indeterminate forms are functions of the form

$$y = [f(x)]^{g(x)}$$

Just as in logarithmic differentiation, we will use the natural logarithm. This allows us to use the properties of logarithms to bring the function in our exponent "down" a level.

Example 19.6: Indeterminate Form 0^0

Evaluate $\lim\limits_{x \to 0^+} x^x$

Solution. Since we have a function in our exponent let $y = x^x$ and apply the natural logarithm to both sides

$$\ln(y) = \ln[x^x] \implies \ln(y) = x\ln(x)$$

Now we take the limit of each side:

$$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0+} x\ln(x)$$

Focusing on the right hand side we re-write this product as

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}}$$

Now the limit has the indeterminate form $\frac{\infty}{\infty}$ so we can apply L'Hôpital's Rule.

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} &\overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[\frac{1}{x}]} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} && \text{Apply L'Hôpital's Rule} \\
&= \lim_{x\to 0^+} \frac{-x^2}{x} = \lim_{x\to 0^+} -x && \text{Rewriting \& Simplifying} \\
&= 0 && \text{By direct substitution}
\end{aligned}$$

So we have

$$\lim_{x\to 0^+} x\ln(x) = 0$$

Since

$$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} x\ln(x) = 0$$

Then we have

$$\lim_{x\to 0^+} \ln(y) = 0$$

We "undo" the $\ln(y)$ by exponentiating each side:

$$\lim_{x\to 0^+} e^{\ln(y)} = e^0 \implies \lim_{x\to 0^+} y = 1$$

Recall that we said that $y = x^x$ and so we have

$$\lim_{x\to 0^+} x^x = 1$$

The cases for 1^∞ and 0^∞ are handled similarly.

In the example above you should note that we used the fact that

$$e^{\left(\lim\limits_{x\to 0^+} \ln(y)\right)} = \lim_{x\to 0^+} e^{\ln(y)} = \lim_{x\to 0^+} y$$

This comes from the fact that we have the limit of a function composition.

> Keep in mind that there are cases in which no amount of manipulation will give us the form we need to use L'Hôpital's Rule. When this occurs, we must use alternative means to find the limit. It is also not appropriate to use L'Hôpital's Rule when using the definition of the derivative. Since L'Hôpital's Rule requires use of the derivative, it makes no sense to use a derivative to evaluate the limit that defines the derivative. For both of these reasons it is still important to learn techniques for evaluating limits that we learned at the beginning of the course.

A word about some textbooks

Some textbooks (not ours) and online systems say to write

$$[f(x)]^{g(x)} = e^{g(x)\ln(f(x))}$$

Where does this come from? Recall that

$$\ln\left(f(x)^{g(x)}\right) = g(x)\ln(f(x))$$

and by expontiating each side we obtain

$$e^{\ln\left(f(x)^{g(x)}\right)} = e^{g(x)\ln(f(x))} \qquad \Longrightarrow \qquad [f(x)]^{g(x)} = e^{g(x)\ln(f(x))}$$

It is preferred that you not skip steps and use memorized formulas to carry out work that can be found just by applying basic properties of logarithms. This property assumes we know what we need to do at the end of the process before we even start. It is not recommended to do things this way. It doesn't help to learn the "tricky tricks" before you even understand how the process itself even works.

The strategy for functions of the form $[f(x)]^{g(x)}$ is the same as for logarithmic differentiation. So it is recommended to just use the same strategy we've seen before and disregard explanations that skip these steps.

Your textbook has limits organized by type in the exercises portion of section 4.8. It is recommend to work through problems from each section so you can see the variety within each type of indeterminate form. It is then recommended that you attempt a random assortment of problems so you can practice recognizing different forms of these limits and applying the relevant strategy without being directed.

Lecture 20

Antiderivatives and Indefinite Integration

Sections
4.10

Antiderivatives

Up to this point in the course it has been assumed that we already know the function whose rate of change is of interest to us. Unfortunately, such an assumption is rarely true for a vast majority of the applications of calculus. It is typically more practical to measure the rate of change of a parameter rather than the parameter itself.

For instance, while driving your car the speedometer tells you how fast you are going but does not directly tell you when you will get to your destination. A biologist may know the rate at which a bacteria population is growing but not the exact number of bacteria in that population. It is easier for a physicist to measure the velocity of a moving particle than its position at specific times. In these cases, the problem is to find a function F whose derivative satisfies the known information. If such a function F exists, it is referred to as the antiderivative of f. The process of finding these functions is referred to as integration.

Definition 20.1: Antiderivative

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Antiderivatives are not necessarily unique. In other words, F is *an* antiderivative and not *the* antiderivative of f. For example the functions

$$F_1(x) = x^4 + 6, \quad F_2(x) = x^4, \quad F_3(x) = x^4 - 17$$

are all antiderivatives of the function $f(x) = 4x^3$.

Theorem 20.1: Representation of Antiderivatives

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on the interval I is of the form

$$F(x) + C$$

for all x in I, where C is a constant.

In other words, you can represent the entire family of antiderivatives of a function by adding a constant to a known antiderivative.

For example, the entire family of antiderivatives for $f(x) = 3x^2$ is given by

$$G(x) = x^3 + C$$

where C is some constant. This C is known as the **constant of integration**. The function G can sometimes be referred to as the **general antiderivative** of f. Sometimes, it is possible to find a *specific* value for the constant C. This is covered later in the section on differential equations.

Notation for Integrals

The operation of finding an antiderivative can be called **antidifferentiation** but is more commonly referred to as **indefinite integration**.

There is specific notation that goes along with integration. Figure 20.1 breaks this notation. The function being integrated is known as the **integrand** while the variable you are integrating with respect to is the **variable of integration**.

The expression $\int f(x)\,dx$ is read as the "integral of f with respect to x".

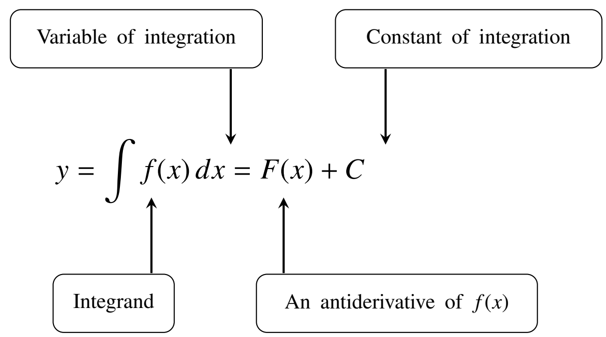


Fig. 20.1: Integral Notation

Basic Integration Rules

Integration can be thought of as the "inverse" or "undoing" of the operation of differentiation. Suppose you take the derivative of a function $f(x)$ and obtain $f'(x)$ then integrating we obtain

$$\int f'(x)\,dx = f(x) + C$$

Due to the "inverse" nature of integration we also have

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

These two equations allow you to obtain integration formulas directly from known differentiation formulas. For example, the most common integration formula you will use is the counterpart to the power rule of differentiation.

Theorem 20.2: Power Rule for Indefinite Integrals

Let p be a real number then

- For $p \neq -1$

$$\int x^p\,dx = \frac{x^{p+1}}{p+1} + C$$

- For $p = -1$

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

where C is an arbitrary constant.

We can also obtain the equivalent basic rules for integrals that we have for derivatives. These are given in Table 20.1 along with the equivalent differentiation rule.

Table 20.1: Basic Integration Rules

Constant Rule

$$\frac{d}{dx}[C] = 0 \implies \int 0\,dx = C$$

Constant Multiple Rule

$$\frac{d}{dx}[kf(x)] = kf'(x) \implies \int kf(x)\,dx = k\int f(x)\,dx$$

Sum & Difference Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x) \implies \int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Example 20.1: Applying Basic Integration Rules

Integrate the following

a.) $\int 6x\,dx$

Solution.

$$\int 6x\,dx = 6\int x\,dx = 6\left(\frac{x^{1+1}}{1+1}\right) + C = \frac{6x^2}{2} + C$$

Note that since C is just an arbitrary constant then $6C$ is also an arbitrary constant. We could call it something different like K but instead just keep using C. In other words, multiplying by 6 will not change the fact that it's an arbitrary constant.

b.) $\int \sqrt{x}\,dx$

Solution.

$$\begin{aligned}\int \sqrt{x}\,dx &= \int x^{1/2}\,dx && \text{Rewrite the integrand}\\ &= \frac{x^{1/2+1}}{1/2+1} + C && \text{Power rule for integrals}\\ &= \frac{2}{3}x^{3/2} + C && \text{Simplify}\end{aligned}$$

c.) $\int \frac{2}{x}\,dx$

Solution.

$$\begin{aligned}\int \frac{2}{x}\,dx &= 2\int \frac{1}{x}\,dx && \text{Constant multiple rule}\\ &= 2\ln x + C && \text{By Theorem 20.2}\end{aligned}$$

In Table 20.2 we summarize some of the common integration formulas you will use.

Table 20.2: Common Integration Formulas

$\int 0\,dx = C$	$\int k\,dx = kx + C$
$\int 1\,dx = x$	$\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$
$\int kf(x)\,dx = k\int f(x)\,dx$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\int e^x\,dx = e^x + C$	$\int e^{ax}\,dx = \dfrac{1}{a}e^{ax} + C$
$\int \dfrac{1}{x}\,dx = \ln\lvert x\rvert + C$	$\int a^x\,dx = \left(\dfrac{1}{\ln a}\right)a^x + C$

Example 20.2: Integrating Simple Functions

Integrate the following functions.

a.) $\int (x^2 + 4x - 7)\,dx$

Solution.

$$\begin{aligned}
\int (x^2 + 4x - 7)\,dx &= \int x^2\,dx + \int 4x\,dx - \int 7\,dx && \text{Sum/Difference rule}\\
&= \int x^2\,dx + 4\int x\,dx - 7\int 1\,dx && \text{Constant Multiple rule}\\
&= \frac{1}{2+1}x^{2+1} + 4\left(\frac{1}{1+1}x^{1+1}\right) - 7x + C && \text{Power rule}\\
&= \frac{1}{3}x^3 + 2x^2 - 7x + C && \text{Simplify}
\end{aligned}$$

b.) $\int \dfrac{x^2 + 3x}{\sqrt{x}}\,dx$

Solution. First we rewrite and simplify the integrand.

$$\begin{aligned}
\int \frac{x^2 + 3x}{\sqrt{x}}\,dx &= \int \left(\frac{x^2}{x^{1/2}} + \frac{3x}{x^{1/2}}\right)dx\\
&= \int \left(x^{3/2} + 3x^{1/2}\right)dx\\
&= \int x^{3/2}\,dx + 3\int x^{1/2}\,dx\\
&= \frac{1}{3/2+1}x^{3/2+1} + 3\left(\frac{1}{1/2+1}x^{1/2+1}\right) + C\\
&= \frac{2}{5}x^{5/2} + 2x^{3/2} + C
\end{aligned}$$

Integration can get complicated pretty quickly. Just as with derivatives, many antiderivatives can only be found be rewriting the integrand into a simpler form that we know how to handle. The following examples are about as complicated as integration will get in this course. Techniques for how to deal with more complicated integrals than these are explored in the second semester of calculus. The one nice thing about integration is that you can always check your work by taking the derivative.

> ⚠ We DO NOT have a direct integration rule for the product or quotient of two functions. Techniques for handling most functions of this form are covered in the second semester of calculus. At this level, you must reduce or simplify your integrand into a form that we can easily find an antiderivative for based on our knowledge of derivatives.

Integrals of Trigonometric Functions

We also have integration formulas for trigonometric functions. Some of the more common integration formulas are given in Table 20.3.

Table 20.3: Integrals of Trigonometric Functions

$\int \sin u\, du = -\cos u + C$	$\int \csc u\, du = -\ln\lvert\csc u + \cot u\rvert + C$
$\int \cos u\, du = \sin u + C$	$\int \sec u\, du = \ln\lvert\sec u + \tan u\rvert + C$
$\int \tan u\, du = -\ln\lvert\cos u\rvert + C$	$\int \cot u\, du = \ln\lvert\sin u\rvert + C$

Example 20.3: Integrating a Trigonometric Function

Integrate $\displaystyle\int \frac{\sin x}{\cos^2 x}\, dx$

Solution. First we rewrite the integrand and apply a simple trig identity to get it in a form we recognize.

$$\int \frac{\sin x}{\cos^2 x}\, dx = \int \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}\, dx = \int \sec x \tan x\, dx$$

Recalling the differentiation rule $\frac{d}{dx}[\sec x] = \sec x \tan x$ we find that

$$\int \frac{\sin x}{\cos^2 x}\, dx = \int \sec x \tan x\, dx = \sec x + C.$$

Introduction to Differential Equations

A **differential equation** is an equation involving an (unknown) function and its derivative. The study of differential equations makes up a large portion of applied mathematics as they are used to model much of the phenomena observed in the real world. The simplest form of a differential equation looks like

$$y'(x) = x^2 + x \quad \text{or} \quad \frac{dy}{dx} = x^2 + x$$

While there are many methods used to obtain solutions to a differential equation, the operation underlying most of them is just plain old integration. By integrating the above equation we obtain

$$y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Since we still have a constant of integration this is known as a **general solution** to the differential equation. In order to find the value of C we need to know the value of our function at a single value of x. This is what is known as an an **initial condition**. Usually, the initial condition is the value of the function at the time $t = 0$ but this is not always the case.

An equation involving derivatives along with an initial condition is what is known as an **initial value problem**. For example, suppose we know that $y(0) = 1$ for the differential equation given above. Then we have the initial value problem

$$y'(x) = x^2 + x, \quad y(0) = 1$$

When we are able to determine the constant of integration we call this a **particular solution** to the differential equation. For example, the above initial value problem above has the particular solution

$$y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 1$$

Example 20.4: Solving an Initial Value Problem

Find the function $y(t)$ that satisfies the initial value problem

$$y'(t) = \frac{3}{t} + 6, \quad y(1) = 8$$

Solution. We know that $\int y'(t)\,dt = y(t) + C$ so to obtain $y(t)$ we first integrate $y'(t)$.

$$y(t) = \int \left(\frac{3}{t} + 6\right) dt = 3\int \frac{1}{t}\,dt + 6\int 1\,dt = 3\ln t + 6t + C$$

This gives us the **general solution** to the differential equation

$$y(t) = 3\ln t + 6t + C$$

To find a particular solution we need to know a value of the function $y(t)$. We are given that $y(1) = 8$ so we plug this into our general solution and solve for C.

$$8 = 3\ln(1) + 6(1) + C \quad \Longrightarrow \; 8 = 0 + 6 + C \quad \Longrightarrow \; C = 2$$

So our **particular solution** is

$$y(t) = 3\ln t + 6t + 2$$

We often have rate of change information for something but need to know the function that governs the rate of change we are observing. This is why the solution of differential equations is such a key part of applied mathematics, engineering, and other sciences. We will demonstrate the solution of these types of problems through some examples.

The most common type of initial value problem you will see at this level is an application involving velocity, acceleration and position. Previously, we were given a position function $s(t)$ and asked to find its velocity,

$v(t)$ or its acceleration $a(t)$. We found these functions by taking the derivative of the position function. In the real world, we actually have the opposite situation. We will have data (and therefore a function) governing the acceleration or velocity and then need to find the position at a particular time.

Why is this? Well, it is much easier to determine how fast an object is going by recording the distance traveled over a certain period of time rather than trying to measure the exact path traveled by the object. Now we will solve a motion problem. In this situation, we will deal with two initial conditions. In general, the number of initial conditions you have for a problem will match the highest order of derivative you have. Recall that $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$.

Example 20.5: Solving a Motion Initial Value Problem

Find the position function given

$$a(t) = 4, \quad v(0) = -3, \quad s(0) = 2$$

Solution. Recall that $a(t) = v'(t) = s''(t)$ so we integrate $a(t)$ to find $v(t)$,

$$v(t) = \int a(t)\,dt = \int 4\,dt = 4t + C$$

The value of the constant C is determined by the initial condition $v(0) = -3$,

$$-3 = v(0) = 4(0) + C \quad \implies C = -3.$$

So our velocity function is

$$v(t) = 4t - 3.$$

To find the position function we recall that $v(t) = s'(t)$ and so

$$s(t) = \int v(t)\,dt = \int 4t - 3\,dt = 4\left(\frac{1}{2}t^2\right) - 3t + D = 2t^2 - 3t + D.$$

We use the initial condition $s(0) = 2$ to solve for D. Note that the use of D for the constant of integration is simply to differentiate it from the previously used C. The letter you choose to represent this constant is essentially arbitrary.

$$2 = 2(0)^3 - 3(0) + D \quad \implies D = 2.$$

So the position function is

$$s(t) = 2t^2 - 3t + 2$$

You may not always have initial conditions with respect to a derivative. This is demonstrated in the following example.

Example 20.6: Solving an Initial Value Problem

Consider the problem addressed in Example 20.5 where $a(t) = 4$ but now with the initial conditions $s(0) = 2$ and $s(2) = 4$.

Solution. In Example 20.5 we found that

$$v(t) = 4t + C.$$

Since we have no other information about $v(t)$ we integrate again to obtain the position function

$$s(t) = \int v(t)\,dt = \int 4t + C\,dt = 2t^2 + Ct + D$$

Now we use the initial conditions $s(0) = 2$ and $s(2) = 4$ to solve for C and D.

$$2 = s(0) = 2(0)^2 + C(0) + D$$
$$4 = s(2) = 2(2)^2 + C(2) + D$$

The first equation tells us that $D = 2$ so we plug this result into the second equation to find

$$2C + 2 = -4 \quad \Longrightarrow \; 2C = -6 \qquad \Longrightarrow \; C = -3.$$

Plugging in $D = 2$ and $C = -3$ we obtain the particular solution

$$s(t) = 2t^2 - 3t + 2$$

Lecture 21

Area Under Curves

Section
5.1

Introduction

When we started our discussion on the derivative we used the tangent line to provide a geometric meaning to its definition. In much the same way, we will use area problems to formulate the idea of definite integration. At first glance, the concept of the indefinite integral and definite integration may seem unrelated to each other but both of these processes are closely related through the Fundamental Theorem of Calculus. Before we can define the definite integral we need to pause for a moment to introduce a concise notation for sums. To be clear, integration is not the only time that you will see summation notation as students! Series play a key role in the solution of differential equations, complexity reduction in nonlinear systems, approximation theory and a variety of other applications.

Sigma Notation

Adding up a bunch of terms can be cumbersome when there are a large number of terms. Luckily, we have notation that will make a sum much easier to handle. This is known as **sigma notation**. The notation is represented by the upper case version of the Greek letter sigma. Although it looks like the letter E in English, it is the equivalent of the letter S in the Greek alphabet to represent the word sum. This notation is a valuable tool as it allows us to express sums in a compact way.

Definition 21.1: Sigma Notation

The sum of n terms $a_1, a_2, a_3 \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the **index of summation** a_i is the **ith term** of the sum, the **upper bound of summation** is n and the **lower bound of summation** is 1. Note that i and n are integers.

Essentially, this notation allows us to represent the a sum of many terms in a repeatable pattern. The index i can be interpreted as the "count" of the term and n the number of terms you want to add up. Both of these values must of course be integers. Also note that the lower bound doesn't have to be 1. In fact, any integer less than or equal to the upper bound is valid.

The index (for example i) in a sum is a dummy variable that serves as a place holder for an actual integer value to be used in that term. Any letter can be used as the index of summation. The letters i, j and k are often used. It is important to recognize that the index of summation does not appear in the terms of the expanded sum.

Example 21.1: Evaluating Basic Summations

Evaluate the following sums

a.) $\sum_{k=1}^{5} k$

Solution. In words this notation tells us "add up the integers from 1 to 5". Note that there is no k in our final result.

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15$$

b.) $\sum_{j=1}^{3}(1 + j^2)$

Solution.

$$\sum_{j=1}^{3}(1 + j^2) = (1 + (1)^2) + (1 + (2)^2) + (1 + (3)^2) = 2 + 5 + 10 = 17$$

c.) $\sum_{n=0}^{4} \sin\left(\frac{n\pi}{2}\right)$

Solution.

$$\begin{aligned}\sum_{n=0}^{4} \sin\left(\frac{n\pi}{2}\right) &= \sin\left(\frac{0\pi}{2}\right) + \sin\left(\frac{1\pi}{2}\right) + \sin\left(\frac{2\pi}{2}\right) + \sin\left(\frac{3\pi}{2}\right) + \sin\left(\frac{4\pi}{2}\right)\\ &= \sin(0) + \sin\left(\frac{\pi}{2}\right) + \sin(\pi) + \sin\left(\frac{3\pi}{2}\right) + \sin(2\pi)\\ &= 0 + 1 + 0 - 1 + 0\\ &= 0\end{aligned}$$

Just as with derivatives and integrals we also have basic rules for a summation. These rules of course deal with multiplying a sum by a constant and the sum and difference of two summations.

Theorem 21.1: Basic Summation Rules

Suppose that $\{a_1, a_2 \ldots, a_k\}$ and $\{b_1, b_2 \ldots, b_k\}$ are two sets of real numbers and that c is also a real number.

- ► Constant Multiple Rule: $\sum_{k=1}^{n} c \cdot a_k = c \cdot \sum_{k=1}^{n} a_k$
- ► Sum and Difference Rule: $\sum_{k=1}^{n}(a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$

Similar to the process of differentiation and integration it is often easier to use the basic summation rules to simplify the process of evaluating a sum. We demonstrate this process in the following examples.

Example 21.2: Using Basic Summation Rules

Evaluate the following summations.

a.) $\displaystyle\sum_{k=1}^{5} 3k$

Solution.

$$\sum_{k=1}^{5} 3k = 3\sum_{k=1}^{5} k = 3(15) = 45$$

b.) $\displaystyle\sum_{j=1}^{3}(j + j^2)$

Solution.

$$\begin{aligned}\sum_{j=1}^{3}(j + j^2) &= \sum_{j=1}^{3} j + \sum_{j=1}^{3} j^2 \\ &= (1 + 2 + 3) + \left(1^2 + 2^2 + 3^2\right) \\ &= 6 + (1 + 4 + 9) \\ &= 6 + 14 \\ &= 20\end{aligned}$$

c.) $\displaystyle\sum_{n=0}^{4} \cos(n\pi) - \sin\left(\frac{n\pi}{2}\right)$

Solution. In a previous example we found that $\displaystyle\sum_{n=0}^{4} \sin\left(\frac{n\pi}{2}\right) = 0$. The Sum and Difference rule lets us use this prior knowledge quickly compute this sum.

$$\begin{aligned}\sum_{n=0}^{4} \cos(n\pi) - \sin\left(\frac{n\pi}{2}\right) &= \sum_{n=0}^{4} \cos(n\pi) - \sum_{n=0}^{4} \sin\left(\frac{n\pi}{2}\right) \\ &= \sum_{n=0}^{4} \cos(n\pi) - 0 \\ &= \cos(0\pi) + \cos(1\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) \\ &= 1 - 1 + 1 - 1 \\ &= 0\end{aligned}$$

What about upper bounds of summation that are very large? It can get tedious very quickly if calculating an even slightly more involved summation. Luckily, for some expressions we have patterns that allow us to develop simple formulas for evaluating certain summations.

Theorem 21.2: Summation Formulas

1. $\sum_{i=1}^{n} c = cn$
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

As we will see in the following examples, the process of evaluating a sum is made significantly easier by combining these known sum formulas with the basic summation rules.

Example 21.3: Evaluating Summations Using Formulas

Evaluate the following summations.

a.) $\sum_{p=1}^{105} p$

Solution.

$$\sum_{p=1}^{105} p = \frac{(105)\big((105)+1\big)}{2} = 5565$$

b.) $\sum_{k=1}^{35} (1+k^2)$

Solution.

$$\sum_{k=1}^{35} (1+k^2) = \sum_{k=1}^{35} 1 + \sum_{k=1}^{35} k^2 = 1(35) + \frac{(35)\big((35)+1\big)\big(2(35)+1\big)}{6} = 14945$$

c.) $\sum_{m=1}^{72} (2m+m^3)$

Solution.

$$\begin{aligned}\sum_{m=1}^{12} (2m+m^3) &= 2\sum_{m=1}^{12} m + \sum_{m=1}^{12} m^3 \\ &= 2\left(\frac{(12)\big((12)+1\big)}{2}\right) + \left(\frac{(12)^2\big((12)+1\big)^2}{4}\right) \\ &= 2\,(78) + (6084) \\ &= 6240\end{aligned}$$

Typically it is rare that formulas like those presented in 21.2 can be obtained for a given series. There is one series for which we do have a formula for obtaining its value. This is the Geometric series.

Definition 21.2: Geometric Series

For real numbers $a \neq 0$ and r a series of the form

$$\sum_{k=0}^{n-1} ar^k = \sum_{k=1}^{n} ar^{k-1}$$

is called a Geometric Series. The general formula for the value of a geometric series is

$$\sum_{k=1}^{n} ar^{k-1} = a\frac{1-r^{n+1}}{1-r}$$

What happens when the upper bound of a summation gets arbitrarily large? The upper bound of a summation does not need to be a finite number. Now that we have some general formulas for sums we might be curious to know the behavior of this sum as more and more terms are added. To accomplish this task we can use our old friend: Limits!

Example 21.4: Evaluating Summations Using Formulas

Evaluate the following summations.

a.) $\sum_{k=1}^{\infty} k$

Solution.

$$\sum_{k=1}^{\infty} k = \lim_{n\to\infty} \sum_{k=1}^{n} k = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$$

b.) $\sum_{k=0}^{\infty} 2^k$

Solution.

$$\begin{aligned}\sum_{k=0}^{\infty} 2^k &= \lim_{n\to\infty} \sum_{k=1}^{n-1} 2^k = \lim_{n\to\infty} \frac{1-2^{n+1}}{1-2} \\ &= \lim_{n\to\infty} \left(2^{n+1}-1\right) \\ &= \lim_{n\to\infty} 2^{n+1} - \lim_{n\to\infty} 1 \\ &= \infty - 1 = \infty\end{aligned}$$

c.) $\sum_{k=0}^{\infty} 2^{-k}$

Solution.

$$\begin{aligned}\sum_{k=0}^{\infty} 2^{-k} &= \lim_{n\to\infty} \sum_{k=1}^{n-1} \left(\frac{1}{2}\right)^k = \lim_{n\to\infty} \frac{1-(\frac{1}{2})^{n+1}}{1-\frac{1}{2}} = \lim_{n\to\infty} \left(2 - 2\left(\frac{1}{2}\right)^{n+1}\right)\\ &= \lim_{n\to\infty} 2 - 2\lim_{n\to\infty}\left(\frac{1}{2}\right)^{n+1}\\ &= 2 - 0 = 2\end{aligned}$$

The Area Problem

We already know several formulas for computing the area of a region in Euclidean geometry. For example, the area of a rectangle is given by $A = bh$ where b and h represent the length of the rectangles base and height respectively. In fact, it is possible to compute the area of any polygon using the area formulas for triangles and rectangles. That is, the area of a polygon can be found by dividing it into triangles and rectangles and adding their areas.

What do we do if we have a shape with curved sides, like a circle? Here is where we would run into problems.

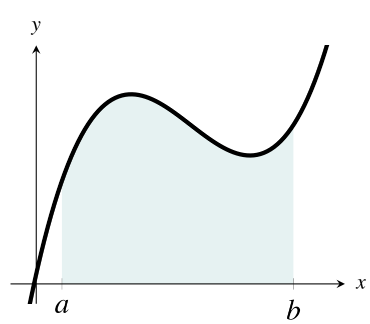


Fig. 21.1: Region under a curve

Our goal in this section is to find a solution to the *Area Problem*: Find the area of the region R that lies under the curve $y = f(x)$ between the boundaries formed from the vertical lines $x = a$ to $x = b$. This is pictured in Figure 21.1.

Recall how we defined the tangent line. We used secant lines to approximate the value of the slope of the tangent line and then took the limit of these approximations. To approximate the area under a curve we will use a similar idea.

We first approximate the area of the region using rectangles. We can get a more accurate approximation with the more rectangles that we use. The exact value for the area under the curve will then be obtained by taking the number of rectangles to infinity via a limit. Unfortunately, this process can be extremely confusing without an appropriate mathematical framework and notation.

Approximating Area by Riemann Sums

In order to approximate an area using rectangles we must first divide the interval from $[a, b]$ into *subintervals*. Each subinterval will define the base of each rectangle used in our approximation. For simplicity we will use subintervals of equal length. This is referred to as a regular partition for the interval $[a, b]$.

Definition 21.3: Regular Partition

A **regular partition** is the division of a closed interval $[a, b]$ into n subintervals of equal length.

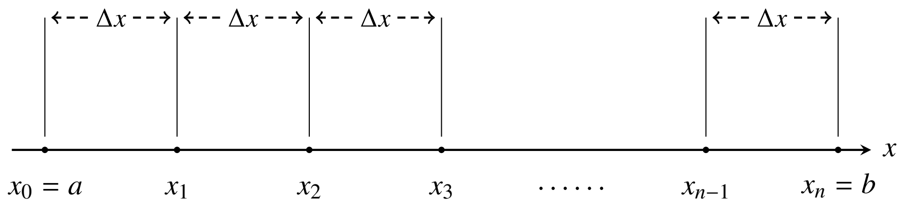


Fig. 21.2: A Regular Partion of $[a, b]$

where

- The **subintervals** are given by

$$[x_0, x_1],\ [x_1, x_2],\ [x_2, x_3],\ \ldots\ ,\ [x_{n-1}, x_n]$$

- The **grid points** are the endpoints of the subintervals given by $x_0,\ x_1,\ \ldots,\ x_n$ where the kth grid point is given by

$$x_k = a + k\Delta x \quad \text{for } k = 0, 1, \cdots, n$$

- The **length** of each subinterval is given by

$$\Delta x = \frac{b-a}{n}$$

Each subinterval of the chosen partition serves as a base for a rectangle used to approximate the area under the curve. That is, we can approximate the area under the curve on each subinterval with the area of a rectangle. The use of a regular partition is convenient for us since the length of each subinterval will be the same (i.e., each rectangle used in our approximation will have the same base length).

The other ingredient we need to define a rectangle is its height. In the kth subinterval $[x_{k-1}, x_k]$, we choose a **sample point** x_k^* and build a rectangle whose height is $f(x_k^*)$, the value of f at x_k^*. So for $b = \Delta x$ and $h = f(x_k^*)$ then the area of the rectangle on the kth subinterval is given by

$$b \cdot h = \Delta x f(x_k^*)$$

The choice of sample point will vary depending on the method we are using. We can see that summing the areas of all of these rectangles gives an approximation of the area under the curve.

Definition 21.4: Riemann Sum on a Regular Partition

Let f be continuous and non-negative on the interval $[a, b]$. For a regular partition of $[a, b]$ into n subintervals an **area approximation** of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots f(x_n^*)\Delta x, \quad x_{i-1} \leq x_i^* \leq x_i$$

where $\Delta x = \frac{(b-a)}{n}$.

This type of area approximation is referred to as a *Riemann sum*. Notice that x_k^* is an arbitrary point on the subinterval $[x_{k-1}, x_k]$. The choice of x_k^* determines the type of Riemann sum being used. Three common choices of x_k^* are x_{k-1}, x_k and $\frac{x_{k-1}+x_k}{2}$. Riemann sums constructed using these three choices are referred to as left, right and midpoint Riemann sums respectively.

Definition 21.5: Left Riemann Sum

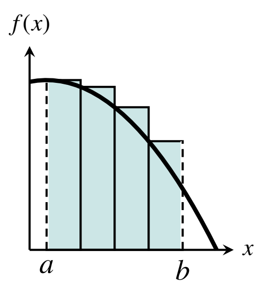


Fig. 21.3
Left Riemann Sum

Let f be continuous and non-negative on the interval $[a, b]$, which is divided into n subintervals of equal length Δx.

A **Left Riemann sum** for f on $[a, b]$ is given by

$$\sum_{i=1}^{n} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + \cdots f(x_{n-1})\Delta x$$

Definition 21.6: Right Riemann Sum

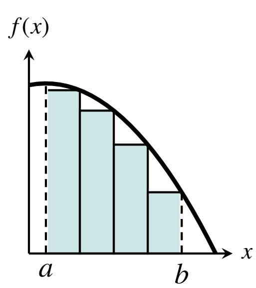


Fig. 21.4
Right Riemann Sum

Let f be continuous and non-negative on the interval $[a, b]$, which is divided into n subintervals of equal length Δx.

A **Right Riemann sum** for f on $[a, b]$ is given by

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots f(x_n)\Delta x$$

Definition 21.7: Midpoint Riemann Sum

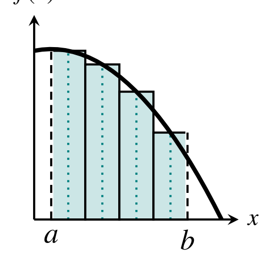


Fig. 21.5
Midpoint Riemann Sum

Let f be continuous and non-negative on the interval $[a, b]$, which is divided into n subintervals of equal length Δx.

A **Midpoint Riemann sum** for f on $[a, b]$ is given by

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots f(x_n^*)\Delta x$$

where $x_i^* = \dfrac{x_{i-1} + x_i}{2}$

Without knowing the shape of the function (or it's exact value) there is no "best" way to choose the point x_k^*. That is, you don't necessarily know which type of Riemann sum is more accurate or easier to use than the others.

Example 21.5: Approximating Area Under a Curve

Use left, right and midpoint Riemann sums with $n = 5$ subintervals to estimate the area of the region lying between the graph of $f(x) = -x^2 + 5$ and the x-axis, between $x = 0$ and $x = 2$.

Compare the value of each approximation to the exact value of the area, $\dfrac{22}{3}$.

Solution. For this example we will use the definition of a regular partition with $a = 0$, $b = 2$ and $n = 5$. Thus, the length of each subinterval is given by

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{5} = \frac{2}{5} = 0.4.$$

The value of the kth grid point is

$$x_k = a + k\Delta x, = 0 + k\left(\frac{2}{5}\right) = k\left(\frac{2}{5}\right) \qquad \text{for} \quad k = 0, 1, 2, 3, 4, 5.$$

In the example we find that

$$\begin{aligned}
x_0 &= 0 + 0\left(\frac{2}{5}\right) = 0, & f(x_0) &= -(0)^2 + 5 = 5\\
x_1 &= 0 + 1\left(\frac{2}{5}\right) = \frac{2}{5}, & f(x_1) &= -\left(\frac{2}{5}\right)^2 + 5 = 4.84\\
x_2 &= 0 + 2\left(\frac{2}{5}\right) = \frac{4}{5}, & f(x_2) &= -\left(\frac{4}{5}\right)^2 + 5 = 4.36\\
x_3 &= 0 + 3\left(\frac{2}{5}\right) = \frac{6}{5}, & f(x_3) &= -\left(\frac{6}{5}\right)^2 + 5 = 3.56\\
x_4 &= 0 + 4\left(\frac{2}{5}\right) = \frac{8}{5}, & f(x_4) &= -\left(\frac{8}{5}\right)^2 + 5 = 2.44\\
x_5 &= 0 + 5\left(\frac{2}{5}\right) = 2, & f(x_5) &= -(2)^2 + 5 = 1.
\end{aligned}$$

The left Riemann sum is given by

$$\sum_{i=1}^{5} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$
$$= 5(0.4) + 4.84(0.4) + 4.36(0.4) + 3.56(0.4) + 2.44(0.4)$$
$$= 8.08.$$

The right Riemann sum is given by

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x$$
$$= 4.84(0.4) + 4.36(0.4) + 3.56(0.4) + 2.44(0.4) + 1(0.4)$$
$$= 6.48.$$

For the function $f(x) = f(x) = -x^2 + 5$ the left Riemann sum was found to be an over-estimate of the the area with about an 10% relative error. The right Riemann sum gave an underestimate of the area with a relative error of about 12%.

Note that we could have done this exercise with the midpoint Riemann sum as well. In fact the midpoint Riemann sum would give an approximate value of 7.36. Try it for yourself!

In the previous example we saw that the resulting approximations where fairly inaccurate. Depending on the setting an approximation with a 10% error could be unacceptable. The primary source of error in that approximation comes from the small number of rectangles used ($n = 5$).

If we wanted a more precise answer we would have needed to use more subintervals to define our partition. However, as we increase the value of n the amount of work required to get the approximation also increases.

If the value of n is large it is helpful to use the known summation formulas for basic sums to simplify the work. In the following examples we will use some of the rules presented in 21.2 to evaluate some Riemann sums with a large value of n.

Example 21.6: Evaluating Riemann Sums When n is Large

Evaluate the left Riemann sums for the region bounded by the graph of $f(x) = x^2$ and the x-axis between $a = 0$ and $b = 2$ using $n = 40$ subintervals.

Solution. With $n = 40$, the length of each subinterval is

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{40} = \frac{1}{20} = 0.05.$$

For the left Riemann sum we choose $x_k^* = x_{k-1}$. So the value of x_k^* in the kth grid point is

$$x_k^* = a + (k-1)\Delta x = 0 + (k-1)0.05 = 0.05k - 0.05 \quad \text{for} \quad k = 1, 2, \cdots, 40.$$

Therefore the left Riemann sum is

$$\begin{aligned}
\sum_{k=1}^{n} f(x_k^*)\Delta x &= \sum_{k=1}^{40} f(0.05(k-1))0.05 \\
&= \sum_{k=1}^{40} (0.05(k-1))^2\, 0.05 \\
&= (0.05)^3 \sum_{k=1}^{40} (k^2 - 2k + 1) \\
&= (0.05)^3 \left(\frac{40(40+1)(2(40)+1)}{6} - 2\left(\frac{40(40+1)}{2}\right) + 40 \right) \\
&= (0.05)^3 (20540) \\
&= 2.5675
\end{aligned}$$

Exact Area

In the next section we will see how Riemann sums are used to define the definite integral. For now we close our discussion about approximate area with the following remark. The accuracy of the area approximation using Riemann sums can be improved by increasing the value of n, the number of intervals used to define the partition of $[a, b]$. It then stands to reason that we could determine the exact value of the area under the curve if an infinite number of rectangles are used in the Riemann sum. We summarize this result in the following definition.

Definition 21.8: Definition of Area Under the Curve

Let f be continuous and nonnegative on the interval $[a, b]$. The **area** of a the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \le c_i \le x_i$$

where $\Delta x = \frac{(b-a)}{n}$.

In the following example we demonstrate how Riemann sums can be used to exactly compute the area under a curve.

Example 21.7: Evaluating Summations Using Formulas

Use left Riemann sums to determine the exact area of the region bounded by the graph of $f(x) = x^2$ and the x-axis between $a = 0$ and $b = 2$.

Solution. To begin, partition $[0, 2]$ into n subintervals each of width

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}.$$

For the left Riemann sum we choose $x_k^* = x_{k-1}$. So the value of x_k^* in the kth grid point is

$$x_k^* = a + (k-1)\Delta x = \frac{2(k-1)}{n} \quad \text{for} \quad k = 1, 2, \cdots, n.$$

Therefore the left Riemann sum is

$$\begin{aligned}
\sum_{k=1}^{n} f(x_k^*)\Delta x &= \sum_{k=1}^{n} f\left(\frac{2(k-1)}{n}\right)\frac{2}{n} \\
&= \frac{2}{n}\sum_{k=1}^{n}\left(\frac{2(k-1)}{n}\right)^2 = \frac{2}{n}\sum_{k=1}^{n}\frac{(2(k-1))^2}{n^2} \\
&= \left(\frac{2}{n}\right)^3 \sum_{k=1}^{n}(k-1)^2 \\
&= \left(\frac{2}{n}\right)^3 \sum_{k=1}^{n} k^2 - 2k + 1 \\
&= \left(\frac{2}{n}\right)^3 \left(\frac{n(n+1)(2n+1)}{6} - 2\left(\frac{n(n+1)}{2}\right) + n\right) \\
&= \frac{4}{3n^3}(2n^3 - 3n^2 + n) \\
&= \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}.
\end{aligned}$$

By taking the limit as $n \to \infty$ of the left Riemann sum we obtain the exact area

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_{k-1})\Delta x = \lim_{n\to\infty} \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2} = \frac{8}{3}.$$

Note that the type of Riemann sums used to find the area under the curve in the previous example does not matter. Instead of left Riemann sums we could have used right or midpoint Riemann sums and arrived at the same answer. Try it! Unfortunately, this property is not true for every function. When the area under the curve of a function is the same regardless of the type of Riemann sum used we say that the function is integrable. In the following lecture we will define the definite integral as the limit of general Riemann sums.

Lecture 22

Definite Integrals and the Fundamental Theorem of Calculus

Sections
5.2.1-5.2.5; 5.3

Introduction

We saw in the previous lecture that a limit of the form

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

arises when computing the area under the curve of a function f. It turns out that this same limit occurs in a wide variety of situations even when f is not necessarily a positive function. In fact limits of this form arise when finding the length of curves, volumes of solids, centers of mass, and work as well as other quantities of interest. Because this type of limit is so important we give it a special name and notation. In the following section we will define the definite integral with a more general form of this limit. Towards the end of this lecture we will see that the operations of differentiation and integration are deeply connected, through the Fundamental Theorem of Calculus.

Definite Integrals

In the previous lecture we discussed Riemann sums with regular partitions. In reality we can define a Riemann sum for any partition of $[a, b]$.

Definition 22.1: General Reimann Sum

Suppose

$$[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n],$$

are subintervals of $[a, b]$ with

$$a = x_0 < x_1 < x_2 < \cdots x_{n-1} < x_n = b.$$

For $k = 1, \cdots, n$ let $\Delta x_k = x_k - x_{k-1}$ be the length of the subinterval $[x_{k-1}, x_k]$ and x_k^* be an arbitrary sample point in $[x_{k-1}, x_k]$. If f is defined on $[a, b]$, the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

is called a **general Riemann sum for f on** $[a, b]$.

We have seen examples where it does not matter what type of Riemann sum we use to define the area under the curve. If the area under the curve of a function will be the same no matter what type of Riemann sum we use we say that that function is integrable. The area under the curve is then given by the **Definite Integral**.

Definition 22.2: Definite Integral

A function f defined on $[a, b]$ is **integrable** on $[a, b]$ if

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists and is unique over all partitions of $[a, b]$ and any choice of sample point x_k*. This limit is called the **definite integral of** f **from** a **to** b, which we write

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Recall that the symbol $\int$ is called an integral sign. In the notation

$$\int_a^b f(x)dx$$

a is the lower limit of integration, b the upper limit of integration, and $f(x)$ is the integrand. The symbol dx has no official meaning by itself. However, it does signal the variable that we are integrating with respect to. It may not seem like an important symbol right now, since the functions we work with are all typically of one variable however, in multivariable calculus this is very important. Recall that the procedure of calculating an integral is called **integration**.

Note the definite integral is a number; it does not depend on x. In fact we could use any letter in its place without changing the value of the definite integral

$$\int_a^b f(x)\,dx = \int_a^b f(r)\,dr = \int_a^b f(t)\,dt$$

As we have already mentioned, the area under the curve of a function can be computed with the definite integral. What about when the graph of the curve does not lie above the x-axis? In this case we must talk about net or signed area. The Riemann sums will then approximate the area of the regions that lie above the x-axis *minus* the area of the regions that lie below the x-axis. In such situations it is possible that the area computed with a definite integral will be negative in value. Since our concept of area requires that it be a positive quantity we use the terms net area or signed area when referring to the area computed with definite integrals.

Definition 22.3: Net (Signed) Area

Consider the region R bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the sum of the areas of the parts of R that lie above the x-axis *minus* the sum of areas of the parts of R that lie below the x-axis on $[a, b]$.

We can now formally state how to find the area of a region using the definite integral in the following theorem.

Theorem 22.1: Definite Integral as Area of a Region

If f is continuous on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Net Area} = \int_a^b f(x)\,dx$$

At long last we have a formal definition of the definite integral! However, this definition comes with a serious constraint. A function is only integrable if this limit exists and is unique for any type of Riemann sum. Just like it was with differentiation it is important to ask ourselves: "What types of functions can we integrate?". The conditions for a function to be *integrable* is given in the following theorem. The proof of these results is beyond the scope of this course.

Theorem 22.2: Integrable Functions

If a function f is continuous on the closed interval $[a, b]$, or if f is bounded on $[a, b]$ with a finite number of discontinuities then f is integrable on $[a, b]$. That is $\int_a^b f(x)\,dx$ exists.

If we know that a function is integrable we can compute the area under its curve using the definition of the definite integral. This process can be tedious but a familiarity with it will help us appreciate the simplicity of the Fundamental Theorem of Calculus.

Example 22.1: Evaluating Definite Integral as a Limit

Using the definition of the Definite Integral evaluate the value of $\int_0^3 x^3 - 6x\,dx$.

Solution. Since $x^3 - 6x$ is continuous on $[0, 3]$ it is also integrable. Thus, the *limit* of any Riemann sum will yield the same value, regardless of our choice of sample points. For this problem I will choose to use right Riemann sums to evaluate the definite integral. For n evenly spaced subintervals we have

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

and

$$x_k = a + k\Delta x = 0 + k\frac{3}{n} = \frac{3k}{n}$$

Using the definition of right Riemann sums we have

$$\begin{aligned}
\int_0^3 x^3 - 6x\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{3k}{n}\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \left[\left(\frac{3k}{n}\right)^3 - 6\left(\frac{3k}{n}\right)\right]
\end{aligned}$$

$$= \lim_{n\to\infty} \frac{3}{n}\sum_{k=1}^{n}\left[\frac{27}{n^3}k^3 - \frac{18}{n}k\right]$$
$$= \lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{k=1}^{n}k^3 - \frac{54}{n^2}\sum_{k=1}^{n}k\right]$$
$$= \lim_{n\to\infty}\left[\frac{81}{n^4}\cdot\frac{n^2(n+1)^2}{4} - \frac{54}{n^2}\cdot\frac{n(n+1)}{2}\right]$$
$$= \lim_{n\to\infty}\left[\frac{81}{4}\left(1+\frac{2}{n}+\frac{1}{n^2}\right) - \frac{54}{2}\left(1+\frac{1}{n}\right)\right]$$
$$= \frac{81}{4} - 27$$
$$= \frac{-27}{4}$$

Since the definite integral is defined as the limit of a sum it inherits a lot of the same properties.

Theorem 22.3: Basic Properties of Definite Integrals

If f and g are integrable on $[a,b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a,b]$, and

- Constant Multiple Rule: $\displaystyle\int_a^b kf\,dx = k\int_a^b f\,dx$
- Sum and Difference Rule: $\displaystyle\int_a^b f \pm g\,dx = \int_a^b f\,dx \pm \int_a^b g\,dx.$

Recall that the the definite integral was defined with the assumption that $a < b$. There are however occasions when it is necessary to reverse the limits of integration ($b < a$), or integrate over a single point ($b = a$). Sometimes it may be necessary to integrate the function separately over parts of the interval ($a < c < b$). This is common for absolute value and piecewise defined functions. To handle these cases we have the following theorem.

Theorem 22.4: Special Properties of Definite Integrals

(i) $\displaystyle\int_a^a f(x)\,dx = 0$

(ii) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

(iii) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

The Fundamental Theorem of Calculus

We have computed definite integrals using the definition as a limit of Riemann sums. We saw that this procedure is sometimes long and difficult. In fact, it is usually not possible or practical. Fortunately, there is a powerful and practical method for evaluating definite integrals. However, in order to understand this method we will need a better understanding of the inverse relationship between differentiation and integration. The bridge we seek to connect these two ideas is known as the Fundamental Theorem of Calculus (FTOC). Since the proof of the FTOC is somewhat complicated, most textbooks present the FTOC in two separate parts. The first part is basically a useful property that can be used to validate the more useful second part.

Theorem 22.5: The Fundamental Theorem of Calculus (Part 1)

If f is continuous on $[a, b]$ then the area function

$$A(x) = \int_a^x f(t)\,dt, \quad \text{for } a \le x \le b$$

is continuous on $[a, b]$ and differentiable of (a, b). The area function satisfies

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Since A is an antiderivative of f on $[a, b]$ it is one short step to a powerful method for evaluating definite integrals. Remember that any two antiderivatives of f differ by a constant. Assuming that F is any antiderivative of f on $[a, b]$ we have

$$F(x) = A(x) + C, \quad \text{for } a \le x \le b$$

Noting that if $x = a$ then we have

$$A(a) = \int_a^a f(t)\,dt = 0$$

and so $A(a) = 0$. It follows that

$$F(b) - F(a) = (A(b) + C) - (A(a) + C) = A(b)$$

Writing $A(b)$ in terms of the definite integral we have the result that

$$A(b) = \int_a^b f(x)\,dx = F(b) - F(a)$$

The result is essentially the second part of the Fundamental Theorem of Calculus, sometimes referred to as the Evaluation Theorem.

Theorem 22.6: The Fundamental Theorem of Calculus (Part 2)

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

We now have a way to compute definite integrals using our knowledge of derivatives. Are you excited? You should be excited! At this point we are free of the dreaded tyranny that is the limit definition of the definite integral. Now if we know the antiderivative of a given function we can easily compute the area under its curve.

Lets pause for a moment to discuss what the fundamental theorem of calculus really means. Part one says that

$$\frac{d}{dx}\int_a^x f(x)\,dx = f(x)$$

and part two says that

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Taken together, the two parts of the Fundamental Theorem of Calculus say that differentiation and integration are an inverse processes i.e. they each undo each other. This realization is essential for our mastery of the process of integration.

Guidelines for Using the Fundamental Theorem of Calculus

- Provided you can find an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
- When applying the Fundamental Theorem of Calculus the following notation is convenient

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

- It is not necessary to include a constant of integration C in the antiderivative because

$$\int_a^b f(x)\,dx = (F(x) + C)\Big|_a^b = \Big[(F(b) + C) - (F(a) + C)\Big] = F(b) - F(a)$$

With the help of the Fundamental Theorem of Calculus we are now capable of evaluating a wide range of definite integrals. We demonstrate this process in the following examples.

Example 22.2: Evaluating a Definite Integral

Evaluate each definite integral.

a.) $\displaystyle\int_1^2 (x^2 - 3)\,dx$

Solution.

$$\int_1^2 (x^2 - 3)\,dx = \left(\frac{1}{3}x^3 - 3x\right)\Bigg|_1^2 = \left(\frac{8}{3} - 6\right) - \left(\frac{1}{3} - 3\right) = \frac{-2}{3}$$

b.) $\displaystyle\int_1^4 3\sqrt{x}\,dx$

Solution.

$$\int_1^4 3\sqrt{x}\,dx = 3\int_1^4 \sqrt{x}\,dx = 3\left(\frac{2}{3}x^{3/2}\right)\Bigg|_1^4 = 2(4)^{3/2} - 2(1)^{3/2} = 14$$

c.) $\displaystyle\int_0^{\pi/4} \sec^2 x\,dx$

Solution.

$$\int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big|_0^{\pi/4} = 1 - 0 = 1$$

It is worth noting that we have only scratched the surface of the study of integration. Just like with differentiation there exists several rules and techniques that are needed to evaluate more complicated integrals. A more complete picture of integration is typically presented in a second semester calculus course.

Lecture 23

Integration Formulas and Applications of Integration

Sections
5.2.6; 5.3.1; 5.4

Introduction

Symmetries are wonderful. Much of mathematics relies on symmetry. In fact, an entire field known as Group Theory is built on the study of symmetry. We will utilize various symmetry properties of functions to assist in evaluating certain integrals. Recall that a function is even if $f(x) = f(-x)$ and odd if $f(-x) = -f(x)$.

Working with Integrals

Theorem 23.1: Evaluating Integrals of Even and Odd Functions

Let a be a positive real number and let f be an integrable function on the interval $[-a, a]$.

1. If f is even, $\displaystyle\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.
2. If f is odd, $\displaystyle\int_{-a}^{a} f(x)dx = 0$.

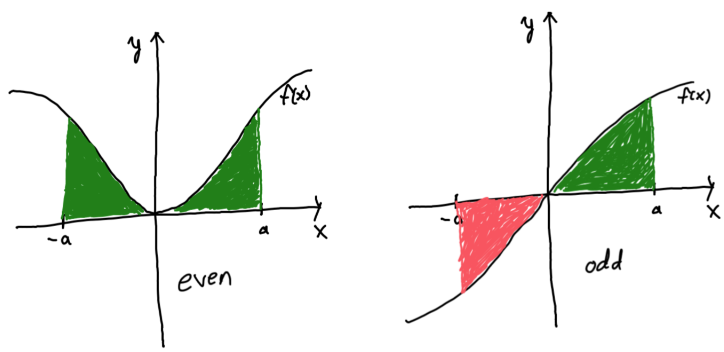


Fig. 23.1: Symmetry and Definite Integrals

Example 23.1: Practice with Symmetry

Verify theorem 22.1 using the following functions:

a.) $f(x) = x^2$

Solution. The function f is even because for all real numbers x,

$$f(-x) = (-x)^2 = x^2 = f(x).$$

We have

$$\int_{-a}^{a} x^2 dx = \frac{x^3}{3}\Big|_{-a}^{a} = \frac{a^3}{3} - \frac{(-a)^3}{3} = \frac{2}{3}a^3$$

and

$$\int_{0}^{a} x^2 dx = \frac{x^3}{3}\Big|_{0}^{a} = \frac{a^3}{3} - \frac{0^3}{3} = \frac{1}{3}a^3,$$

so it is clear that theorem 22.1 works for $f(x) = x^2$.

b.) $g(x) = x^3$

Solution. The function g is odd because for all real numbers x,

$$g(-x) = (-x)^3 = -x^3 = g(x).$$

Because

$$\int_{-a}^{a} x^3 dx = \frac{x^4}{4}\Big|_{-a}^{a} = \frac{a^4}{4} - \frac{(-a)^4}{4} = \frac{a^4}{4} - \frac{a^4}{4} = 0,$$

theorem 22.1 also works for $g(x) = x^3$.

Example 23.2: More Symmetry Practice

Evaluate the following integrals using symmetry arguments.

a.) $\int_{-2}^{2} (x^4 - 3x^3) dx$

Solution. Despite the fact that x^4 is even and $3x^3$ is odd, their difference is neither even nor odd. However, we can divide and conquer:

$$\begin{aligned}\int_{-2}^{2} (x^4 - 3x^3) dx &= \int_{-2}^{2} x^4 dx - 3\int_{-2}^{2} x^3 dx \\ &= 2\int_{0}^{2} x^4 dx - 0 \\ &= 2\left(\frac{x^5}{5}\right)\Big|_{0}^{2} = 2 \cdot \frac{32}{5} = \frac{64}{5}\end{aligned}$$

b.) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x - 4\sin^3 x) dx$

Solution. The expression $\cos x$ is even. The term $\sin x$ is odd, and when raised to an odd power – e.g. $\sin^3 x$ – results in an odd term. Therefore

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x - 4\sin^3 x)dx &= 2\int_0^{\frac{\pi}{2}} \cos x dx - 0 \\ &= 2\sin x|_0^{\frac{\pi}{2}} = 2(1-0) = 2\end{aligned}$$

Definition 23.1: Average Value of an Integrable Function

The average value of an integrable function f on the interval $[a, b]$ is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx$$

Note that this is sometimes denoted as $⨍ f(x)dx$.

Remark: The definition stems from an attempt to generalize the concept of arithmetic mean of a set of numbers to a function. Realizing that the average value of certain functions do not make sense (e.g. average value of $\frac{1}{x}$ on $[0, 1]$), we focus on integrable functions.

Let $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ be any partition of the interval $[a, b]$ into n subintervals each with length $\Delta x_k = x_k - x_{k-1}$ for $k = 1, 2, \cdots, n$, then select an x_k^* on each $[x_{k-1}, x_k]$. The average function values, i.e. $f(x_k^*)$, weighted by the width of the k-th subinterval, i.e. Δx_k, is

$$\frac{\sum_{k=1}^n f(x_k^*)\Delta x_k}{\sum_{k=1}^n \Delta x_k}.$$

Note that both the numerator and the denominator are general Riemann sums. Let $\Delta = \max\{\Delta x_1, \Delta x_2, \cdots, \Delta x_n\} \to 0$, then

$$\begin{aligned}\lim_{\Delta\to 0} \frac{\sum_{k=1}^n f(x_k^*)\Delta x_k}{\sum_{k=1}^n \Delta x_k} &= \frac{\lim_{\Delta\to 0}\sum_{k=1}^n f(x_k^*)\Delta x_k}{\lim_{\Delta\to 0}\sum_{k=1}^n \Delta x_k} \\ &= \frac{\int_a^b f(x)dx}{\int_a^b dx} \\ &= \frac{1}{b-a}\int_a^b f(x)dx\end{aligned}$$

The process above is one of many equivalent ways to discover definition 22.1.

Example 23.3: Average Value of a Function on an Interval

Find the average value of $f(x) = \sin x$ on $[0, \pi]$.

Solution.

$$\bar{f} = \frac{1}{\pi - 0} \int_0^{\pi} \sin x dx = \frac{1}{\pi} \cdot (-\cos x)|_0^{\pi} = \frac{2}{\pi}.$$

We end this section with an integral analogy to the Mean Value Theorem.

Theorem 23.2: Mean Value Theorem for Integrals

Let f be continuous on the interval $[a, b]$. There exists a point c in (a, b) such that

$$f(c) = \bar{f} = \frac{1}{b - a} \int_a^b f(t)dt.$$

Proof. Apply the mean value theorem on $F(x) = \int_a^x f(t)dt$, which is continuous on $[a, b]$ and differentiable on (a, b). □

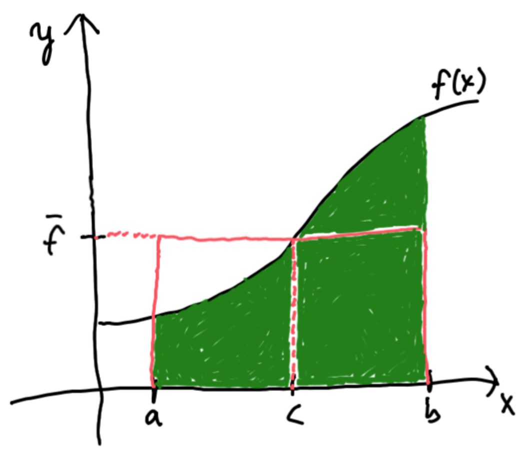


Fig. 23.2: MVT for integrals

In Calculus II, you will learn more universal techniques that are used to evaluate integrals. Now, let us loop back to the position and velocity problem.

Applications of Integration

Velocity and Net Change

Remark: Suppose an object moves along a line with velocity $v(t)$. Then

- The **position** at time t is $s(t)$ such that $s'(t) = v(t)$.
- The **displacement** of the object from $t = a$ to $t = b$ is

$$s(b) - s(a) = \int_a^b v(t)dt.$$

- The **distance traveled** by the object from $t = a$ to $t = b$ is

$$\int_a^b |v(t)|dt,$$

where the integrand $|v(t)|$ is the speed of the object at time t.

Suppose an object moves along a line with acceleration $a(t)$. Then

- The **velocity** at time t is $v(t)$ such that $v'(t) = a(t)$.
- The **change in velocity** of the object from $t = a$ to $t = b$ is

$$v(b) - v(a) = \int_a^b a(t)dt.$$

Example 23.4: Find Velocity and Position Given Acceleration and Initial Condition

Given $a(t) = 10$ as the acceleration function, $v(0) = -10$ as the initial velocity, and $s(0) = 0$ as the initial position, find $v(t)$ and $s(t)$.

Solution.

$$v(t) = v(0) + \int_0^t a(t)dt = v(0) + \int_0^t 10dt = -10 + 10t,$$
$$s(t) = s(0) + \int_0^t v(t)dt = s(0) + \int_0^t (-10 + 10t)dt = 5t^2 - 10t.$$

The End!