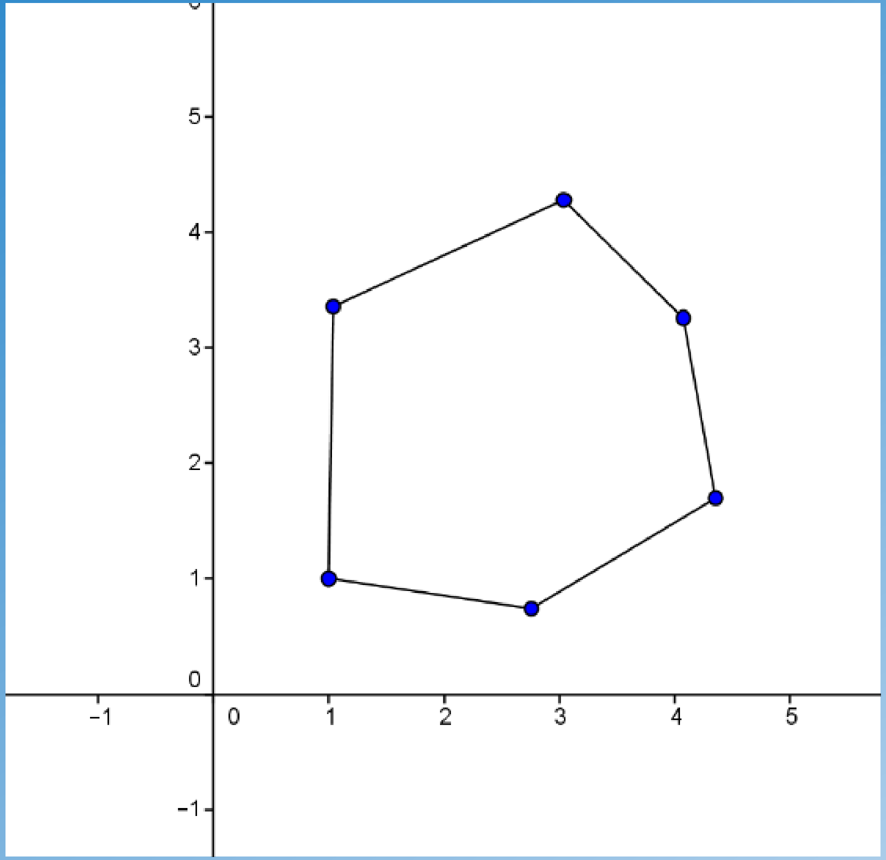
Lattice-point enumeration in polytopes:

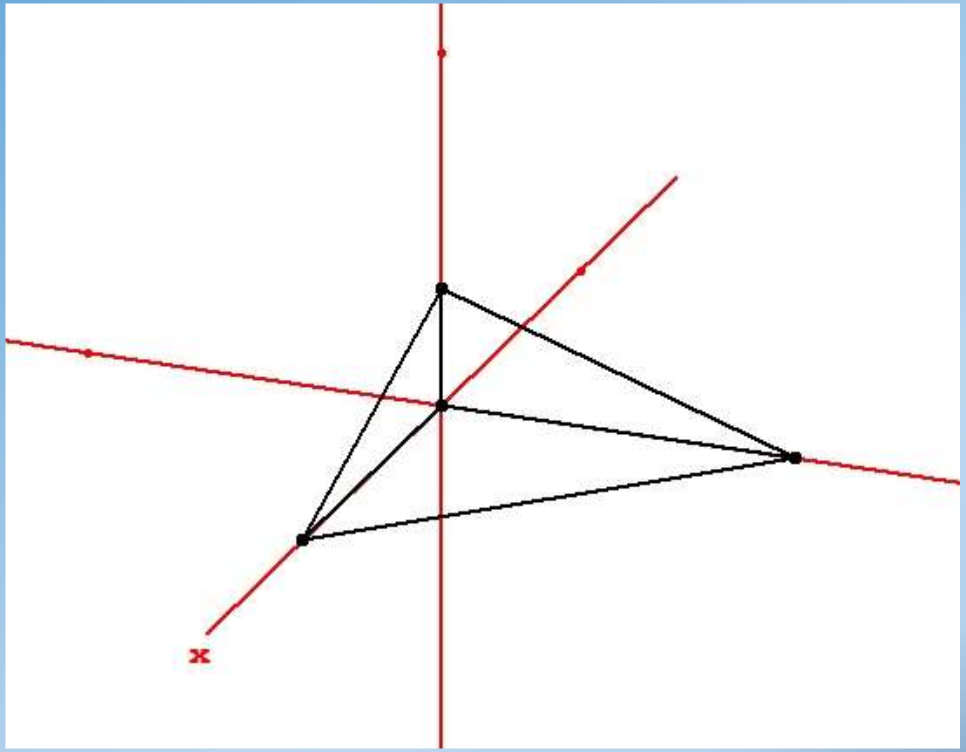
Study of the coefficients of the Ehrhart quasi-polynomial

Outline:

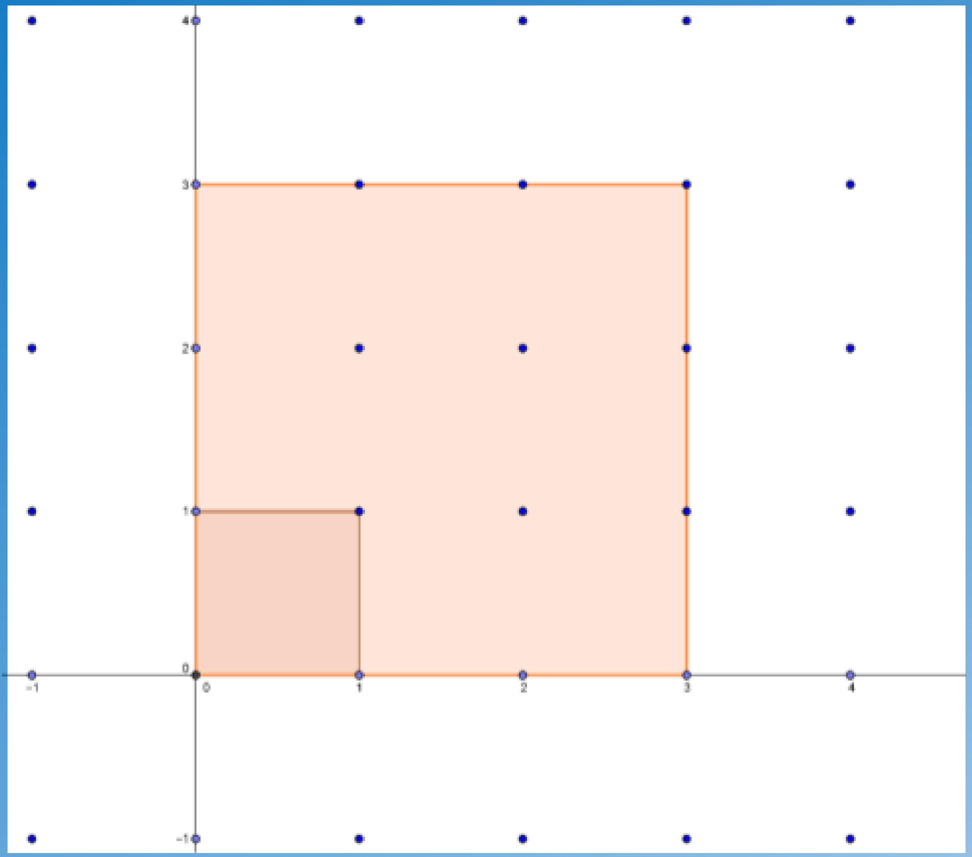
1. Polytopes, convex hull
2. Why do we care?
3. Counting lattice points in polytopes
4. The periods of the Ehrhart quasi polynomial

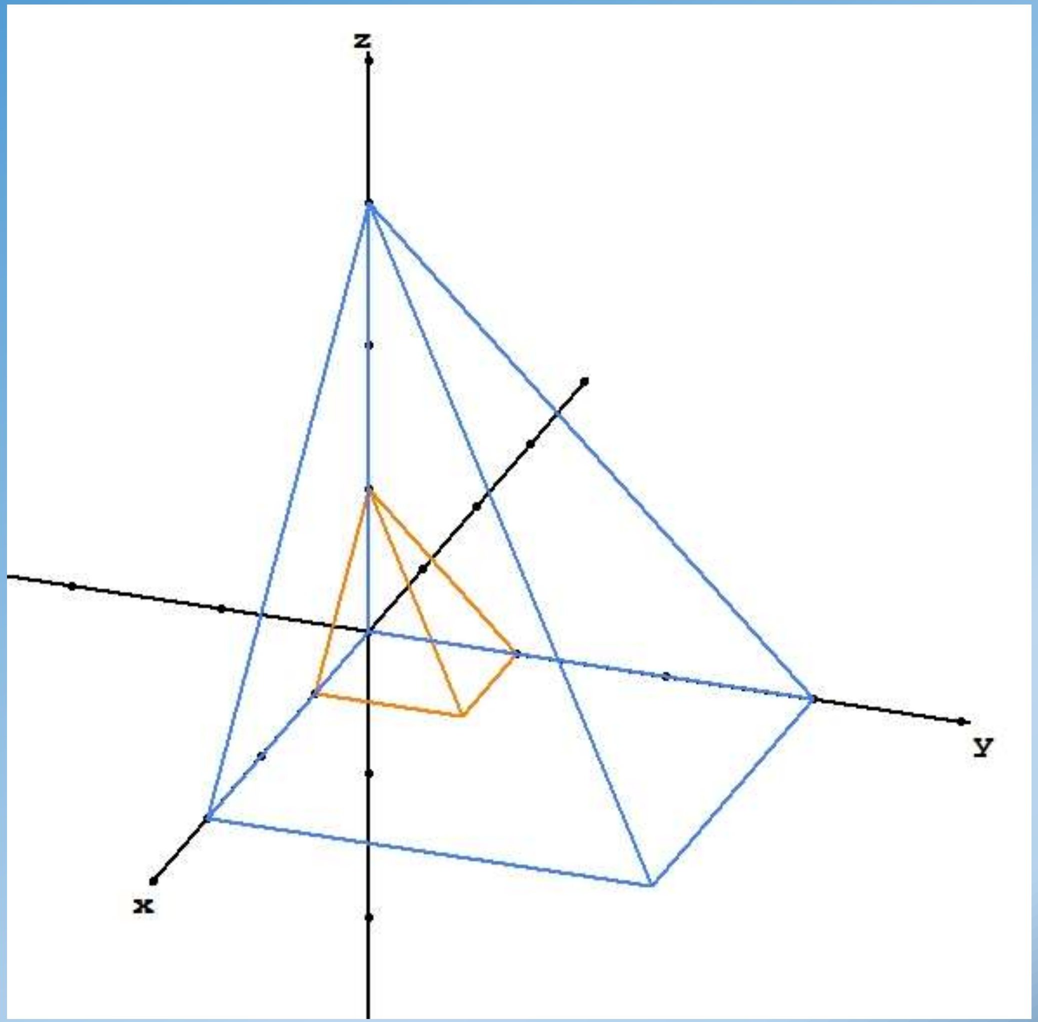
1. Polytopes, convex hull





Integer dilates of polytopes: family of polytopes

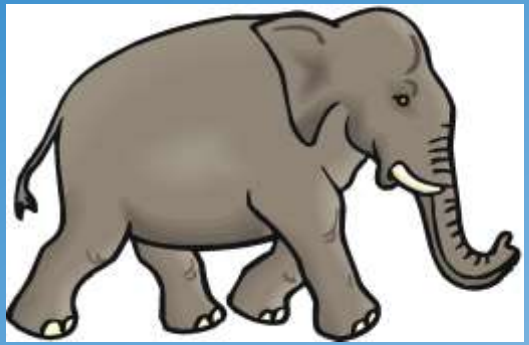




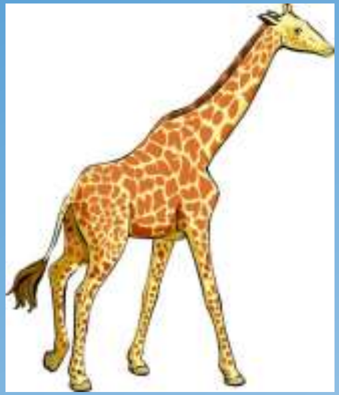
The unit square and its 3rd dilate. If we let P be the unit square (side length of 1), then $3P$ is the square of side length of 3.

2. Why do we care?

- A lot of problems boil down to finding integer solutions to a problem or knowing if there is a solution at all.
- A typical example:
  Suppose we want to load elephants and giraffes on a plane



3,000 kg



1,000 kg

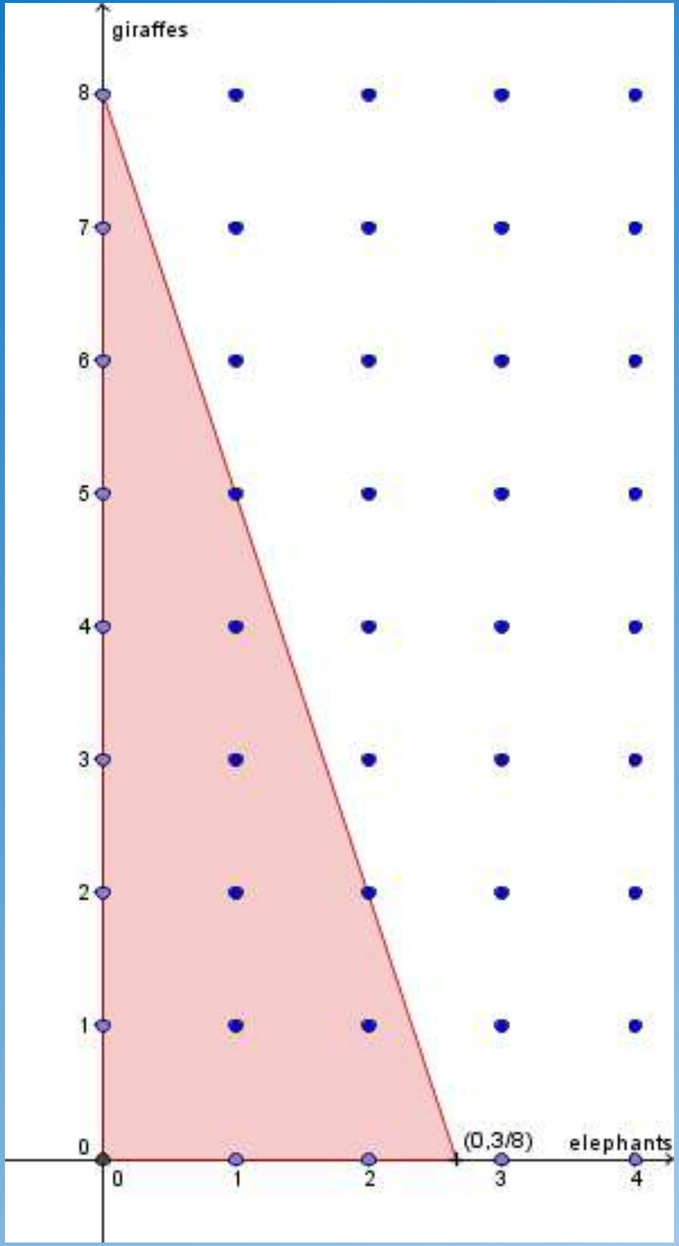
8,000 kg

How many elephant and giraffes can we put on the plane? Or, how many integer solutions do we have to :

$$3{,}000 * E + 1{,}000 * G \leq 8{,}000$$

where $E, G \geq 0$ **and <u>integers</u>!**

- Geometrically:



In general we have:

$$k_1x_1 + k_2x_2 + \cdots + k_jx_j \leq n$$

where

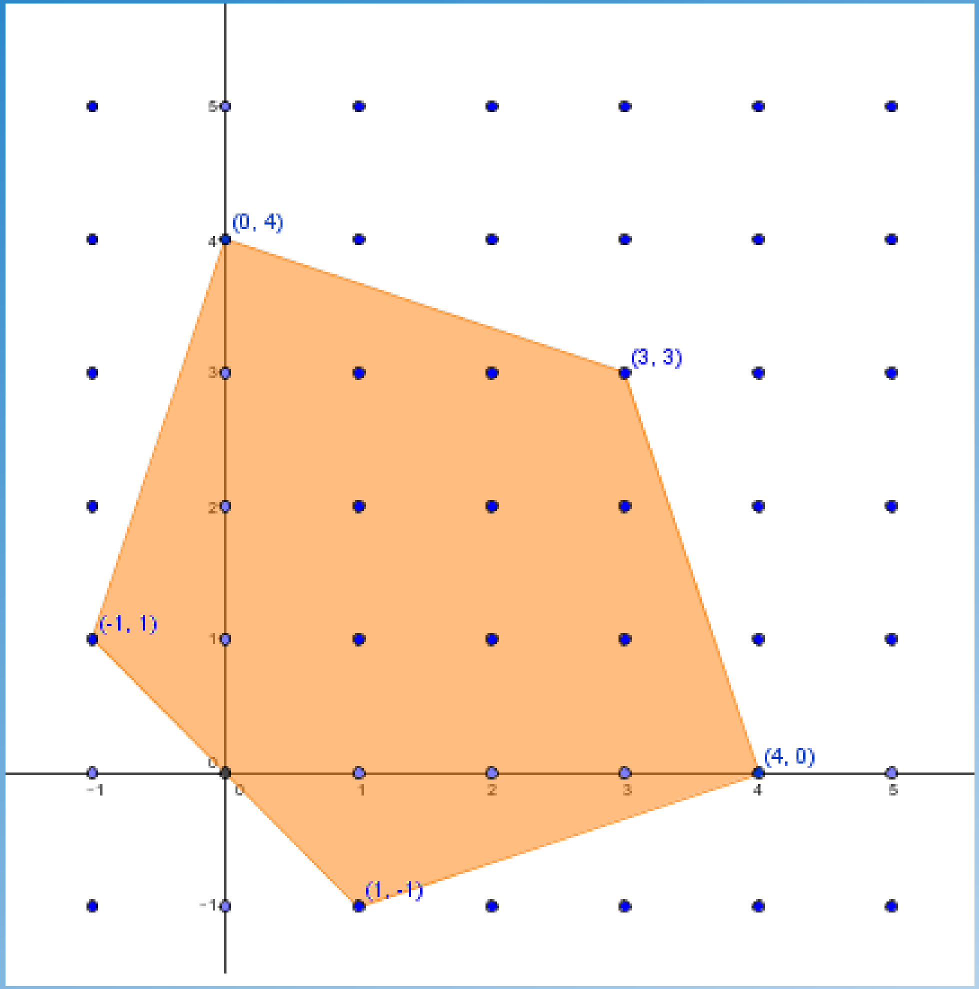
x_i = number of animals of type i

k_i = weight of an animal of type i

n = weight capacity of plane

3. Counting lattice points in polytopes

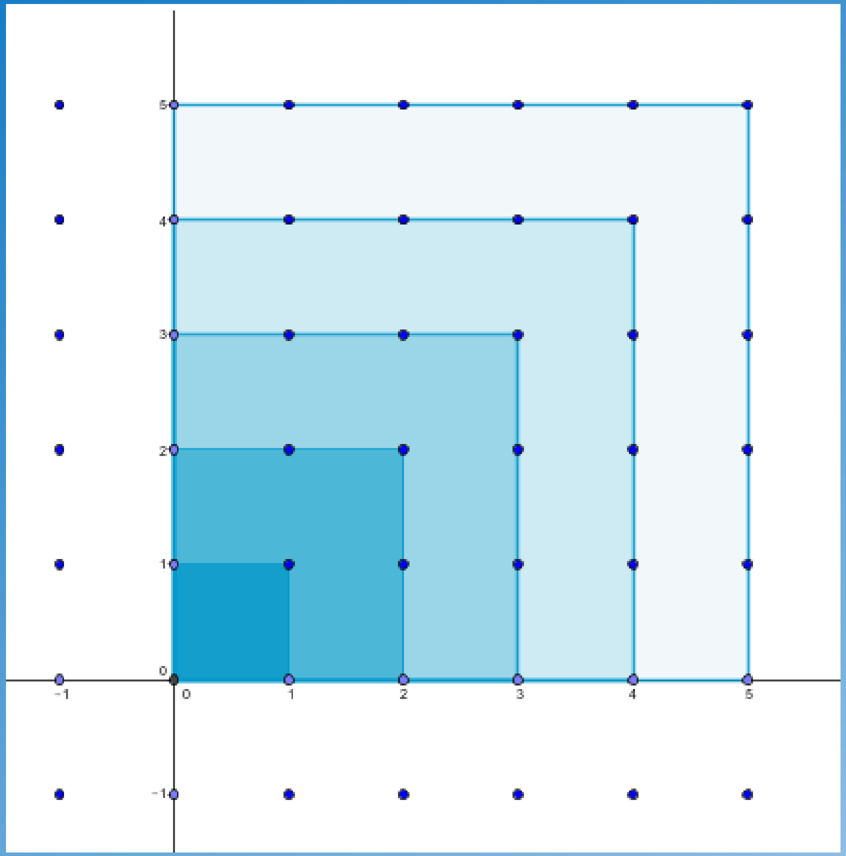
a. Integer polytopes



- The vertices have all integer coordinates
- **<u>The Ehrhart function:</u>**

  $ehr_P(n) =$ number of lattice points in nP

- Example



Let P be the unit square.

We have:

$ehr_P(1) = \#\ of\ lattice\ points\ in\ the\ unit\ square\ C = 4$

$ehr_P(2) = \#\ of\ lattice\ points\ in\ 2C = 9$

$ehr_P(3) = \#\ of\ lattice\ points\ in\ 3C = 16$

$ehr_P(4) = 25$

$ehr_P(5) = 36$

In general,

$ehr_P(n) = (n+1)^2 = n^2 + 2n + 1$

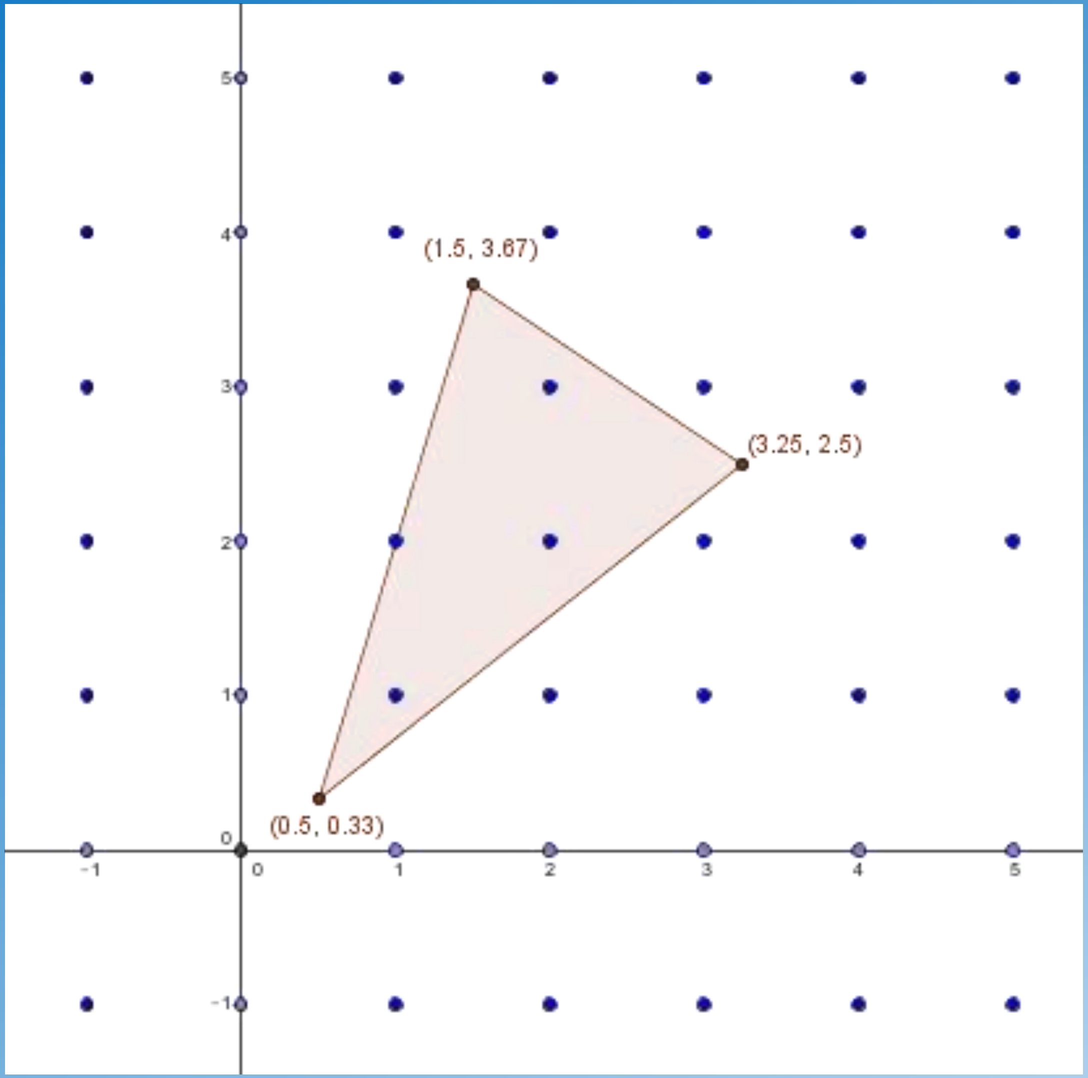
<u>Theorem:</u>

If P is an integer polytope, then $ehr_P(n)$ is a polynomial in n. That is,

$$\boldsymbol{ehr_P(n) = a_t n^t + a_{t-1} n^{t-1} + \cdots + a_1 n + a_0}$$

for some constants $a_0, a_1, \ldots, a_{t-1}, a_t \in \mathbb{R}$.

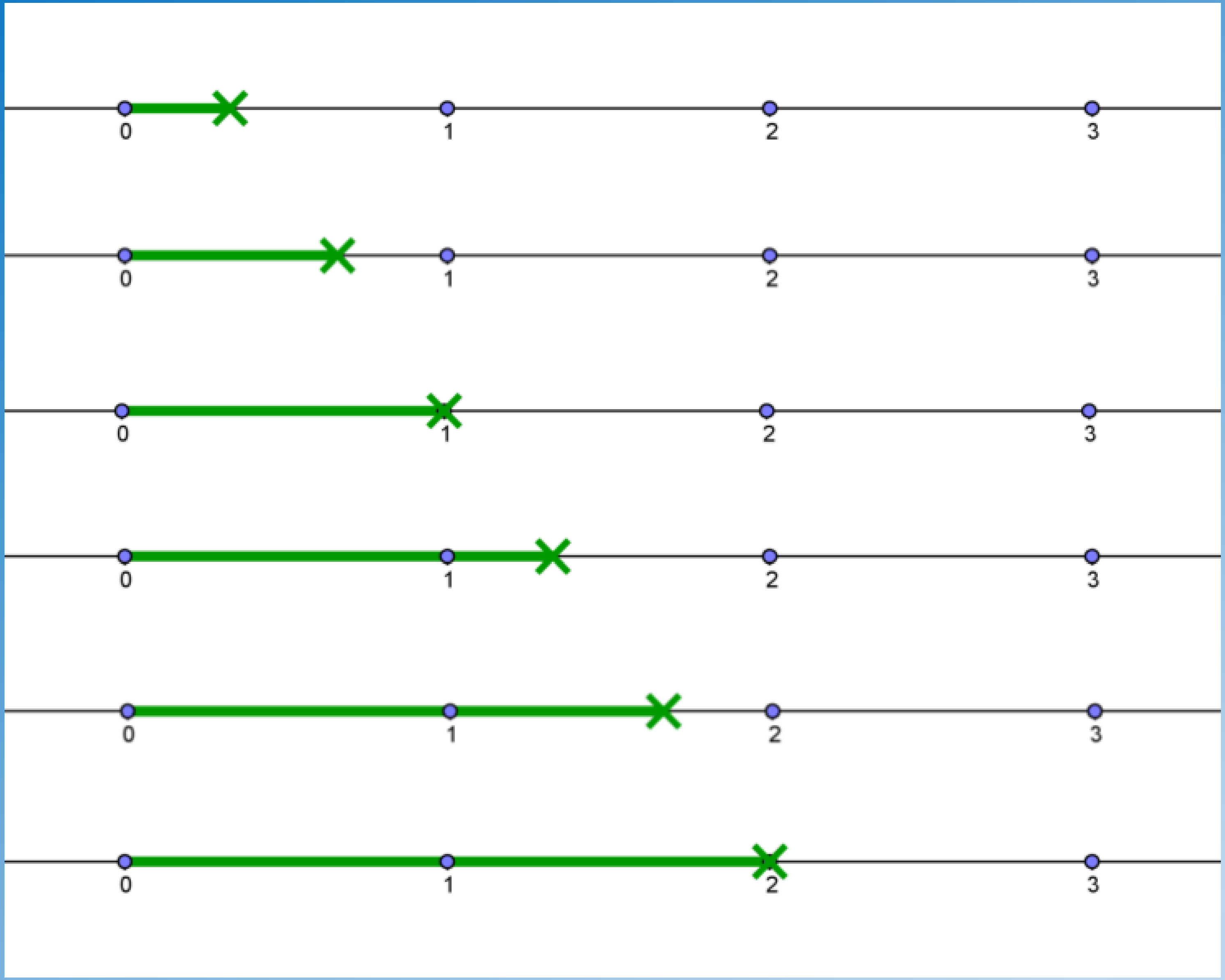
b. Rational polytopes



- The vertices have all rational coordinates. That is, they can be written as fractions.

- Again,
  $ehr_P(n)$ = number of lattice points in nP

- Example



Let l be the segment $[0, \frac{1}{3}]$. Then

$$ehr_l(n) = \left\lfloor \frac{n}{3} \right\rfloor + 1$$

where $\lfloor x \rfloor$ is the biggest integer smaller or equal to x. Then $\lfloor 3.5 \rfloor = 3$ and $\lfloor 3 \rfloor = 3$.

Then, $ehr_l(n) = \frac{1}{3}n + a(n)$

where $a(n) = \left\lfloor \frac{n}{3} \right\rfloor - \frac{n}{3} + 1$

Or, $a(n) = \begin{cases} 1 \text{ if } n = 3k \\ \frac{2}{3} \text{ if } n = 3k + 1 \\ \frac{1}{3} \text{ if } n = 3k + 2 \end{cases}$

Theorem:

If P is an rational polytope, then $ehr_P(n)$ is a quasi-polynomial in n. That is,

$$ehr_P(n) = a_t(n)n^t + a_{t-1}(n)n^{t-1} + \cdots + a_1(n)n + a_0(n)$$

for some periodic functions $a_0(n), a_1(n), \dots, a_{t-1}(n), a_t(n)$

Definition

In $\mathbb{R}^3$ we say that P has period sequence (s_3, s_2, s_1, s_0) if the minimum period of the coefficients $a_3(n), a_2(n), a_1(n), a_0(n)$ is s_i for $i = 0, 1, 2, 3$, where

$$ehr_P(n) = a_3(n)n^3 + a_2(n)n^2 + a_1(n)n + a_0(n)$$

Theorem:

In $\mathbb{R}^3$, $s_3 = 1$ for any polytope P. In general, in $\mathbb{R}^t$, $s_t = 1$ and $a_t(n) = constant$.

4. The periods of the Ehrhart quasi-polynomials

a. Results!

Theorem (Rochais, 2015)

- **Given positive integers s and t, there exists a convex polytopes P with period sequence $(1, 1, s, t)$.**

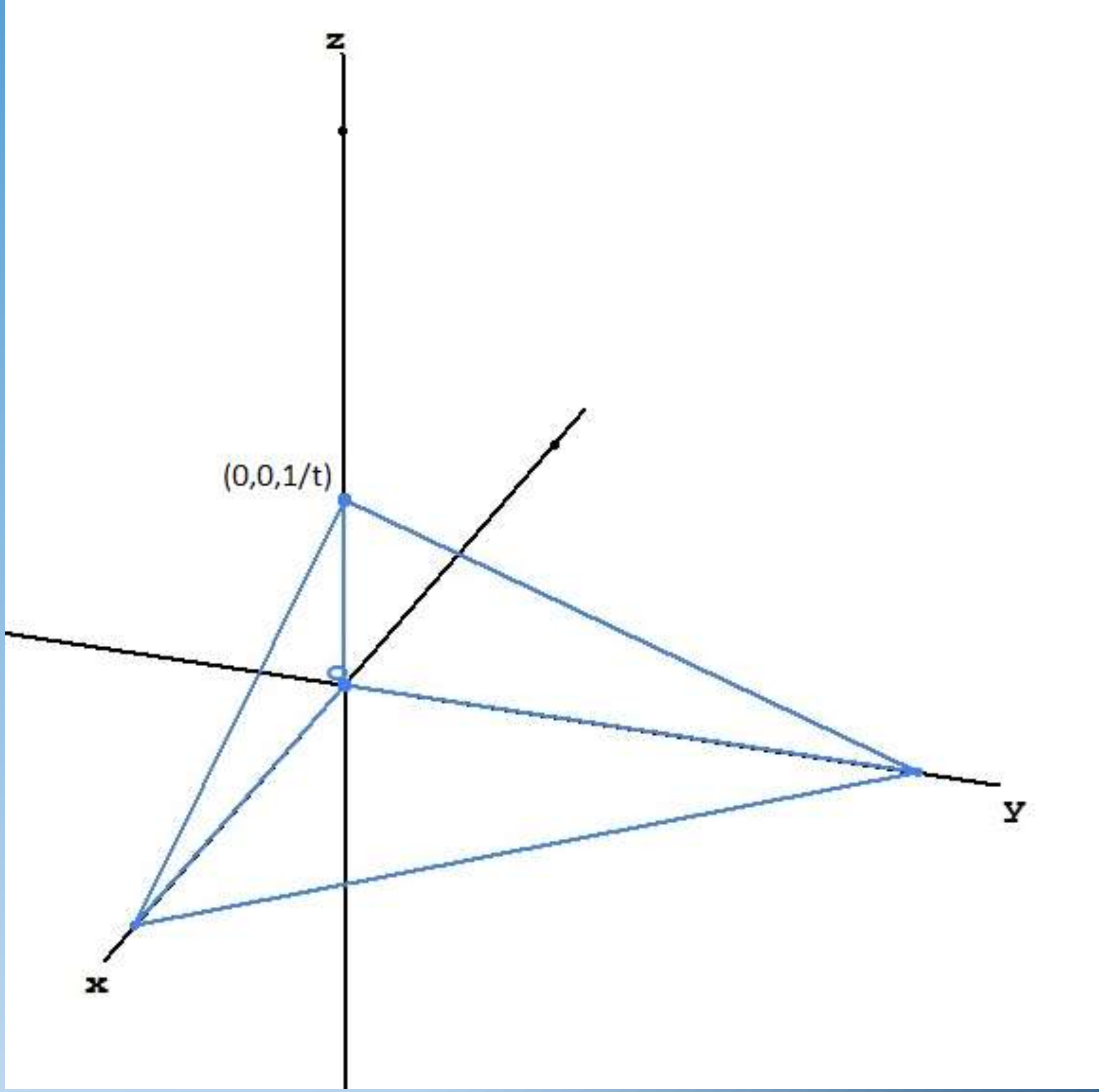
- **Given positive integers s, t and u, there exist a non-convex polytope H with period sequence $(1, u, s, t)$.**

b. A polytope of period sequence $(1, 1, 1, t)$

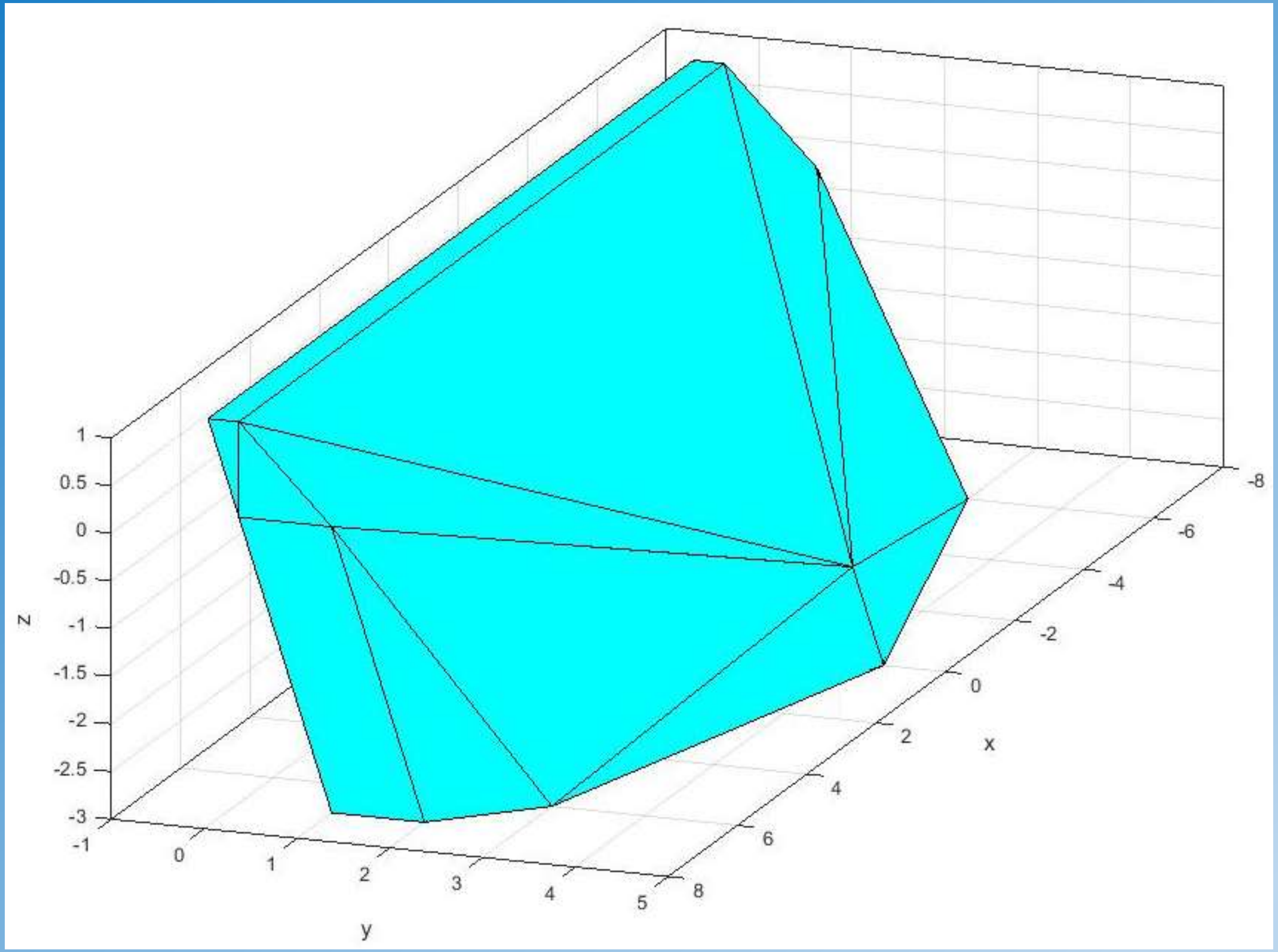
Let

$$P = conv\{(0,0,0), (1,0,0), (0,1,0), \left(0,0,\frac{1}{t}\right)\}$$

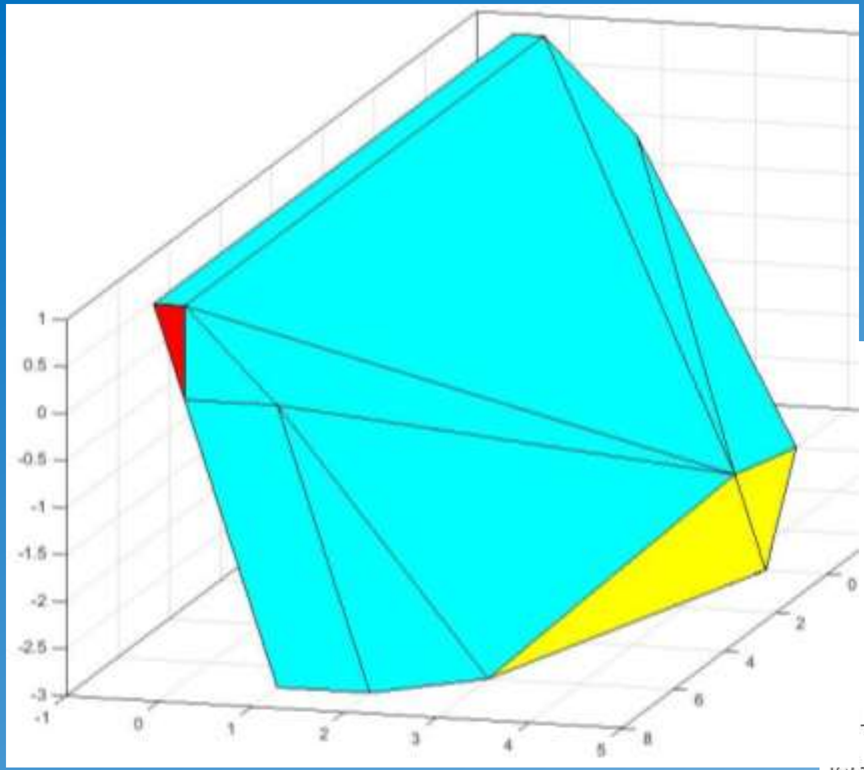
Then P has period sequence $(1,1,1,t)$.

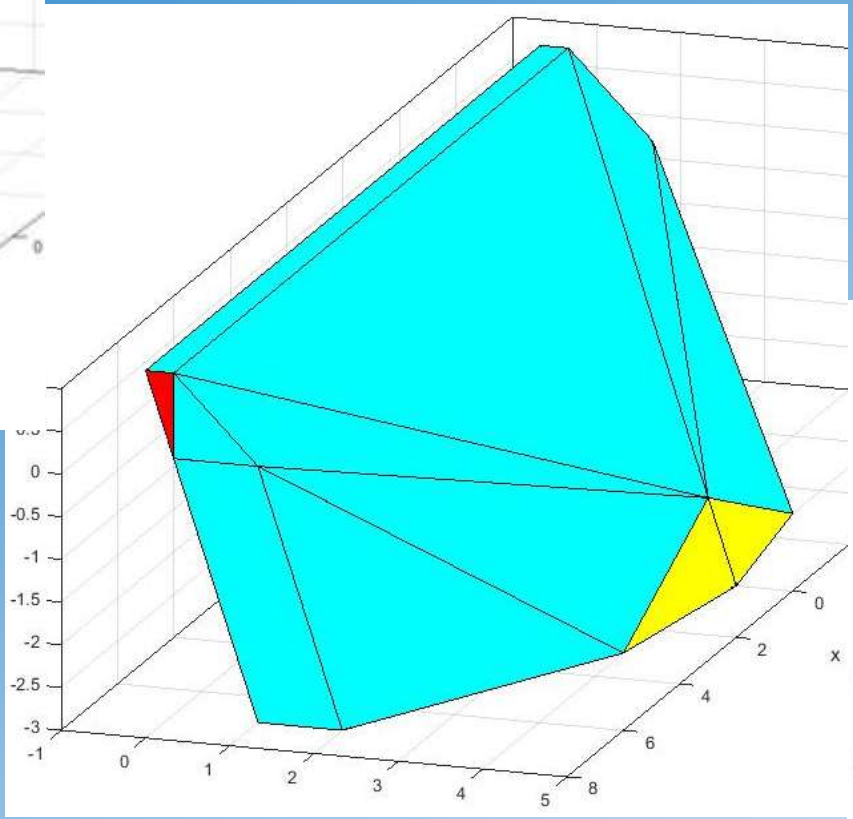


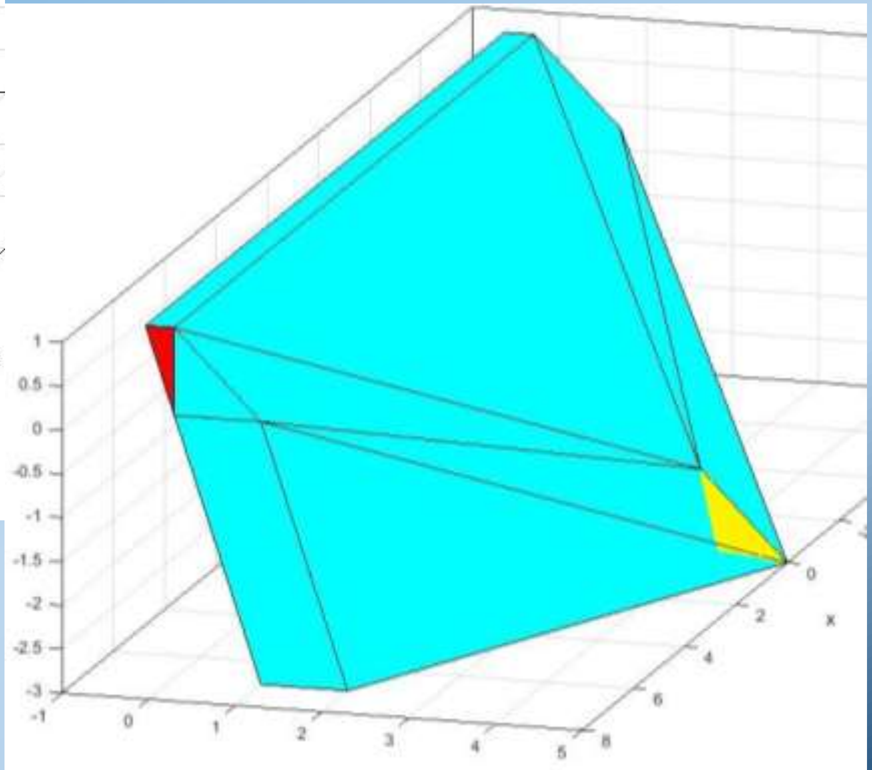
c. A polytope of period sequence $(1, 1, s, 1)$ in 3 dimensions



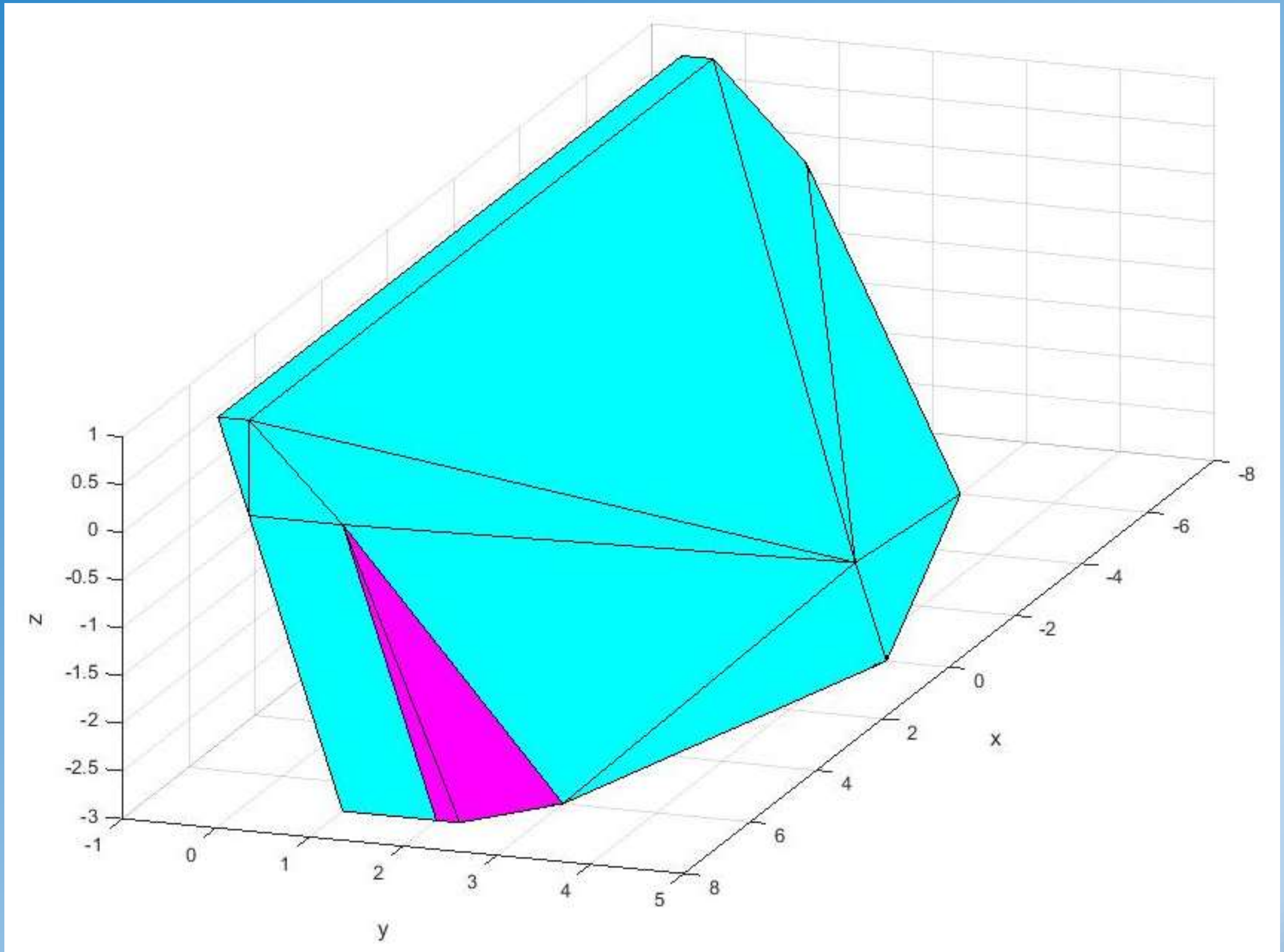
A polytope of period sequence $(1, 1, s, 1)$



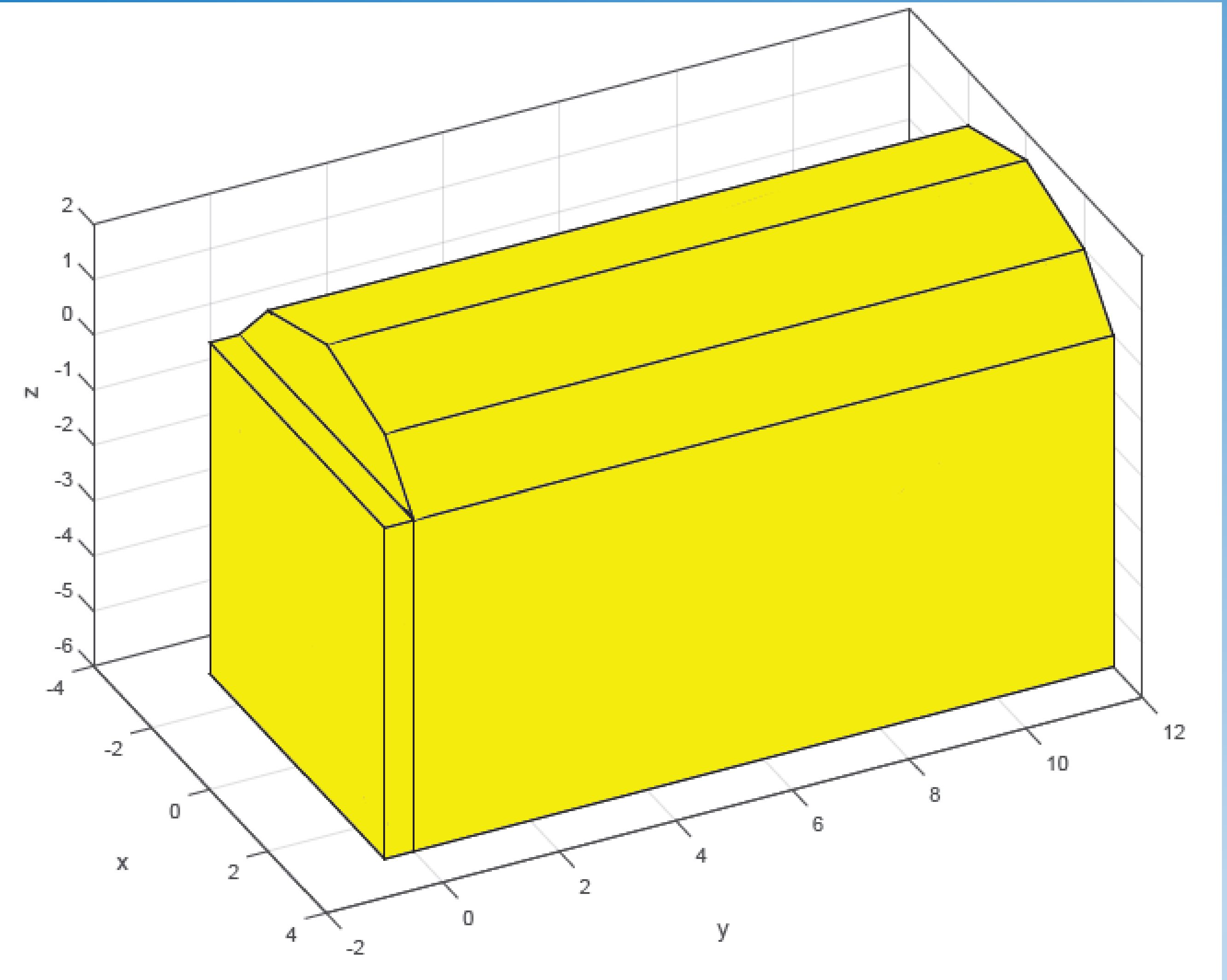




b. Combining $(1, 1, 1, t)$ with $(1, 1, s, 1)$ to get a polytope with period sequence $(1, 1, s, t)$

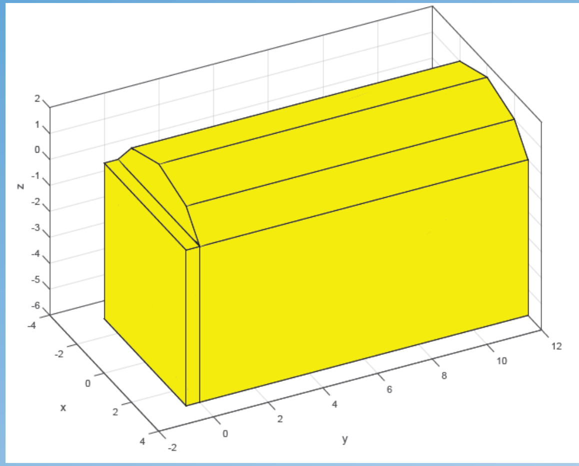


d. A non convex polytope of period $(1, u, 1, 1)$ in 3 dimensions



6. What's next:

- Try to come up with a convex example of a polytope of period $(1, u, 1,1)$
- Generalize to higher dimensions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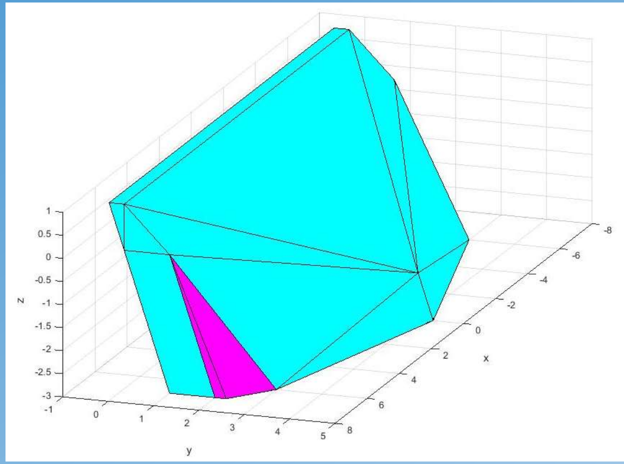
2 1 0 -1 -2 -3 -4 -5 -6
x
-4 -2 0 2 4
y
-2 0 2 4 6 8 10 12

Questions?


z
1 0.5 0 -0.5 -1 -1.5 -2 -2.5 -3
y
-1 0 1 2 3 4 5
x
-8 -6 -4 -2 0 2 4 6 8

Skew Unimodular Transformations

We use piece-wise affine unimodular transformations as a way to skew polytopes without changing the lattice so leaving the number of lattice points inside a polytope intact. This is a very useful way to geometrically manipulate a polytope without changing its Ehrhart polynomial or quasi-polynomial.