Non-transitivity and Probability

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Background

- Humans use words such as: "is taller than", "is smarter than", "is faster than" to describe relationships between objects.
- One convenient property of a relationship is transitivity:

If a is related to b, and b is related to c, then a is related to c.

- Many relations are naturally transitive (e.g. "is taller than", "<", "divides")
- But some relationships are non-transitive: that is, there are objects a, b and c with a related to b, and b related to c, but a is not related to c.
- A simple example of non-transitivity is rock-paper-scissors. Rock beats scissors, scissors beats paper, but rock does not beat paper! In fact paper beats rock.
- Transitivity allows one to draw conclusions, and so often we believe that transitivity holds. Indeed, we consider it an "upset" (i.e. something unexpected) when Wyoming beats CSU, and CSU beats Utah, but Wyoming doesn't beat Utah.

Rolling die

- Dice A has sides 1, 4, 7
- Dice B has sides 2, 5, 8
- Dice C has sided 3, 6, 9.

Two die are selected and rolled. The dice with the largest number wins.

So

  - C beats B 6 out of 9 times
  - B beats A 6 out of 9 times
  - So how about C versus A?

The non-transitive dice game

- One example of non-transitive probability is in non-transitive dice.
- Dice A: 3, 3, 5, 5, 7, 7
- Dice B: 2, 2, 4, 4, 9, 9
- Dice C: 1, 1, 6, 6, 8, 8
- Dice A beats dice B with a 5/9ths edge. Dice B beats dice C with a 5/9ths edge. Dice C beats dice A with a 5/9ths edge. Thus, A beats B, and B beats C.

  But A doesn't beat C!

  We have a non-transitivity.

Purpose

- The purpose of my research was to investigate and construct certain types of non-transitive dice.
- I investigated into a three-person dice game, as well as a n-person dice game.
- I defined an n-person dice game as a non-transitive dice game such that if n-1 of the players pick dice from a pool, then the nth player can always pick one of the remaining dice to have the largest probability of winning among all n players.

Number of dice

- To start my research, I investigated into how many dice would be needed for a 3-person game.
- With four dice, (A, B, C, and D) I assumed with no loss of generality that C wins in the game ABC, thus AB can be responded to with C. AC could not be responded to with B, so it would require D. BC similarly could not be responded to with A, so it would require D. Thus D wins ACD and BCD. Thus, CD has no response, as D has an edge no matter what player 3 chooses.
- With five dice, however, a three-person non-transitive dice game could work. An example of win-response set is on the following slide.

Example win-response set

- ABC: C wins
- ABD: B wins
- ABE: A wins
- ACD: D wins
- ACE: C wins
- ADE: A wins
- BCD: B wins
- BCE: E wins
- BDE: E wins
- CDE: D wins

- AB: respond C
- AC: respond D
- AD: respond B
- AE: respond C
- BC: respond E
- BD: respond E
- BE: respond A
- CD: respond B
- CE: respond D
- DE: respond A

Number of dice, generalized

- The previous example works because the number of three-dice matches is equal to the number of two-dice situations.
- This result can be generalized, for an n-person game.
- Suppose there are 2n-1 dice, and n persons. There are 2n-1 choose n-1 "situations" (sets of n-1 dice) that each require a "response" dice such that the response dice wins the "match" (set of n dice).
- The number of sets of n dice among the 2n-1 dice is 2n-1 choose n. Since 2n-1 choose n equals 2n-1 choose n-1, the number of matches is equal to the number of responses, and every situation requires a unique response.
- Thus the number of matches must equal or exceed the number of situations. So a n-person game requires at least 2n-1 dice.

Five dice

- In order to obtain my set of five dice, I started with a set of five dice that was a successful 2-person game.
- A: 4, 4, 4, 4, 4
- B: 3, 3, 3, 3, 3
- C: 2, 2, 2, 7, 7
- D: 1, 1, 6, 6, 6
- E: 0, 5, 5, 5, 5

- This set did not work, as AB had two responses (which will not work for five dice and three people) and CD and CE had no response. I went through several such failed sets.

- A: 4, 4, 4, 4, 4
- B: 3, 3, 3, 3, 8
- C: 2, 2, 2, 7, 7
- D: 1, 1, 6, 6, 6
- E: 0, 5, 5, 5, 5

- This set did not work either, however it let me to notice that I was getting closer, and the pattern of A having 0 big, 5 small, B having 1 big, 4 small, C having 2 big, 3 small, and so on, with the larger the number of small sides, the larger the values of a dice's "big" and "small". This idea, with some tweaks, lead to my next dice set.

- A: 5, 5, 5, 5, 5, 10
- B: 1, 6, 6, 6, 6, 6
- C: 2, 2, 7, 7, 7, 7
- D: 3, 3, 3, 8, 8, 8
- E: 4, 4, 4, 4, 9, 9

- This set did not work either, however, with a few more tweaks, it lead to my working three-person, five-dice set.

Five dice, revised

- This set of five dice does work for a three-person game.
- A: 5, 5, 5, 5, 8, 12
- B: 0, 6, 6, 6, 6, 10
- C: 1, 4, 7, 7, 7, 7
- D: 2, 2, 2, 9, 9, 9
- E: 3, 3, 3, 3, 11, 11
- If players one and two select one dice each, player three can always pick a dice with an edge in the match. In addition, A beats B, B beats C, C beats D, D beats E, and E beats A. So this set is also a two-person non-transitive dice game set.

Further Research

- Other results
  - Dice A can beat dice B and dice C in two-dice matchups, but B or C could still have the edge in a three-dice matchup.
  - Dice for any non-transitive game cannot have only two sides.
  - Dice for a three-person non-transitive game must have more than three sides.
- Open problems:
  - Determine the minimum number of sides in a three-person, five-dice non-transitive game.
  - Determine whether or not there exist four-sided and five-sided dice sets for a three-person game.
  - Determine the minimum sides for an n-person game.
  - Additional dice. Can we always add an additional dice to a non-transitive die set, to get a non-transitive die set?

Questions?