Analysis of Solutions of 2nd Order Stochastic Parabolic Equations.

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SYNOPSIS

- Stochastic Parabolic Equations.
  - Linear Deterministic Equation.
  - Linear Stochastic Equation.
  - Karhunen-Loève Expansion.
  - Algorithm of Zhang and Lu.
- Computational Results.
  - Description of Numerical Scheme.
  - Code Description.
  - Effects of Varying Sigma.
  - Effects of Varying K.

Linear Deterministic Equation

- Below is shown the linear deterministic equation considered in this project.

$$(1)\quad \begin{cases} u_t - \nabla \cdot (a(x)\nabla u) = f(x,t), & x \in X \subset\subset \mathbb{R}^d, \quad t \in (0,T) \\ u(x,0) = u_0(x), & x \in X \\ u(x,t) = 0, & x \in \partial X \end{cases}$$

- Note that only one spatial dimension, *x*, is considered along with a time dimension, *t*.
- This problem will have a real solution and can be solved deterministically; that is to say *a(x)* and *f(x,t)* will be some functions with no randomness (stochasticity).
- Also note *f(x,t)* doesn't depend on solution *u(x,t)*.

Linear Stochastic Equation

- Randomness is introduced through a random variable, *ω*, into *a(x).*

$$(2)\quad \begin{cases} u_t - \nabla \cdot (a(x;\omega)\nabla u) = f(x,t), & x \in X \subset\subset \mathbb{R}^d, \quad t \in (0,T) \\ u(x,0) = u_0(x), & x \in X \\ u(x,t) = 0, & x \in \partial X \end{cases}$$

where $a(x;\omega) = e^{Y(x;\omega)}$ and $Y(x;\omega)$ is a random process.

- A deterministic method may no longer be used to solve this problem due to the stochasticity introduced by *ω.*

Karhunen-Loève Expansion

- The method used in this project to solve these stochastic parabolic differential equations is called the Karhunen-Loève expansion.

- In order to better understand the Karhunen-Loève expansion, a covariance function must first be defined.

Covariance Function

- A covariance function measures the strength of the mutual dependence of values of a random process at different points.
- The covariance function used in this project is shown below.

$$(3)\ C_Y(x, y) = \langle (Y(x) - \langle Y(x) \rangle\ )(Y(y) - \langle Y(y) \rangle\ ) \rangle$$

where *Y* is some random process.

Karhunen-Loève Expansion

- Now let *Y(x; ω)* be some random process on the unit interval I=[0, 1] with some expectation $\langle Y(x) \rangle$.
- Then according to the theorem of Karhunen and Loève, the following approximation formula can be obtained.

$$(4)\ Y(x;\omega) = \langle Y(x) \rangle + \sum_{i=1}^{\infty} \sqrt{\lambda_i} f_i(x) \xi_i(\omega)$$

where $\{\xi_i\}$ are independent standard normal variables

Karhunen-Loève Expansion

- $\{\lambda_i, f_i\}$ are eigenvalue-eigenvector pairs of the following integral operator

$$(5)\ \mathcal{I}[f](x) = \int_I C_Y(x,y) f(y) dy$$

- The functions f_i are chosen such that the following condition is satisfied.

$$||f_i|| = 1 \text{ in the } L^2 \text{ norm}$$

- Also note that the Karhunen-Loève expansion is optimal in L^2, or every truncated finite sum in the right-hand side represents the best finite-dimensional approximation of Y in L^2.

Description of Algorithm of Zhang and Lu

- Consider the linear stochastic equation presented previously, (2). Let $\langle Y \rangle = 0$ and Var $Y = \sigma^2$ where Var is used to represent variance.

- Since $\{\xi_i\}$ are standard normal variables, the following approximation can be made.

$$(6) Y(x;\omega) \approx \sum_{i=1}^{K} \sqrt{\lambda_i} f_i(x) \xi_i(\omega)$$

- From this point forward, $\sqrt{\lambda_i}$ will be absorbed into $f_i(x)$ since they always appear together in the eigenfunction expansions.

- Now, due to stochasticity, there is no deterministic solution to be found. Assume that the solution $u(x,t;\omega)$ can be expressed in the following manner as a probabilistic series expansion.

- It is important to note that in the preceding equation we assume the following rate of dependence on m.

$$u^{(m)} \sim \sigma^m$$

$$Y^m \sim \sigma^m$$

Description of Algorithm of Zhang and Lu

- The solution can be expressed in the following form as a probabilistic series expansion.

$$(7)\ u(x,t;\omega) = u^{(0)}(x,t) + u^{(1)}(x,t;\omega) + u^{(2)}(x,t;\omega) + \cdots$$

- Substituting the Taylor expansion for e^{Y} into (7) and (2) and equations are grouped according to order with respect to σ, we obtain equations of the following form.

$$\mathcal{L}u^{(0)} = f(x,t) \qquad \text{Order 0}$$

$$\mathcal{L}u^{(1)} = \nabla \cdot (Y\,\nabla u^{(0)}) \qquad \text{Order 1}$$

$$\mathcal{L}u^{(2)} = \nabla \cdot (Y\,\nabla u^{(1)}) + \frac{1}{2}\,\nabla \cdot (Y^2 \nabla u^{(0)}) \qquad \text{Order 2}$$

$$\text{where } \mathcal{L} = \frac{\partial u}{\partial x} - \frac{\partial^2}{\partial x^2}$$

Description of Algorithm of Zhang and Lu

- Notice that in this group of equations, Order 0 has no stochasticity and can be solved deterministically.

- However, something else will have to be done for the higher orders. Substituting what was obtained in (6) for Y and utilizing the fact that $u^{(1)}$ can be expanded as a sum with respect to $\{\xi_i\}$, the stochasticity can be eliminated and what is left is a system of deterministic equations

$$\mathcal{L}u_i^{(1)}(x,t) = \nabla \cdot (f_i \nabla u^{(0)}), \qquad i = 1, 2, \ldots, K$$

Description of Algorithm of Zhang and Lu

- While a similar process can be used for subsequent orders, there will be multiple indices for orders higher than 1.
- For Order 2 there will be indices *i* and *j* etc.
- If the right hand side is replaced by the average of every possible permutation, the right hand side will result in the exact same equation independently of the order in which indices are considered.

Description of the Numerical Scheme

- First, a more detailed description of how to solve parabolic equations is necessary.
- The stiffness matrix A_{ij} and load vector F_i must be obtained.
- To do this, we must define a set of basis functions.
- A basis function is a piecewise real-valued function which spans two adjacent elements and whose maximal value is one.

Description of the Numerical Scheme

- The basis function used is of the following form.

$$\varphi_i(x) = \begin{cases} \frac{1}{n-1}x - \frac{1}{n-1}(x_{i-1}) & x_{i-1} < x < x_i \\ \frac{1}{n-1}x + \frac{1}{n-1}x_{i+1} & x_i < x < x_{i+1} \end{cases}$$

where n is the number of nodes on the mesh.

- If $\{x_i\}$ are nodes on the unit interval, then the elements of the stiffness matrix are defined by the following formula.

$$A_{ij} = \int_{x_{i-1}}^{x_{i+1}} a(x)\varphi`_i(x)\varphi`_j(x)dx$$

where $\varphi`_i(x)$ and $\varphi`_j(x)$ are the first derivatives of the basis functions

Description of the Numerical Scheme

- To obtain the value F_i, of the load vector corresponding to the node x_i we use the following formula.

$$F_i = \int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)dx$$

- Note that the goal of discretization in these formulas is to obtain a discrete approximation of the solution to (1). This approximation is as follows.

$$U(x,t) = \sum_{j=1}^{F} \alpha_j(t)\varphi_j(x)$$

Description of the Numerical Scheme

- Now, substituting this expression for the solution into (1), multiplying both sides by $\varphi_i(x)$, and integrating with respect to *x*, a vector ODE can be obtained for the approximation of $U(x,t)$.

$$(8)\ D\acute{\alpha}(t) + A(t)\alpha(t) = F(t)$$

- D is the identity matrix, A is the stiffness matrix, and F is the load vector.

Description of the Numerical Scheme

- Equation (7) is then solved using an already established scheme, the Crank-Nicolson method.
- The terms of the Karhunen-Loève expansion are computed as follows. Consider the following covariance function.

$$C_Y(x,y) = \sigma^2 e^{-\frac{(x-y)^2}{\eta^2}}$$

- The integration in (5) is performed over an interval $[x_i, x_{i+1}]$ yielding a matrix of values.
- Next, eigenvalue/eigenvector pairs are obtained using a predefined function from the Eigen library.
- The results here were plugged into an existing framework from graduate advisor Kevin Lenth to obtain results.

Code Description

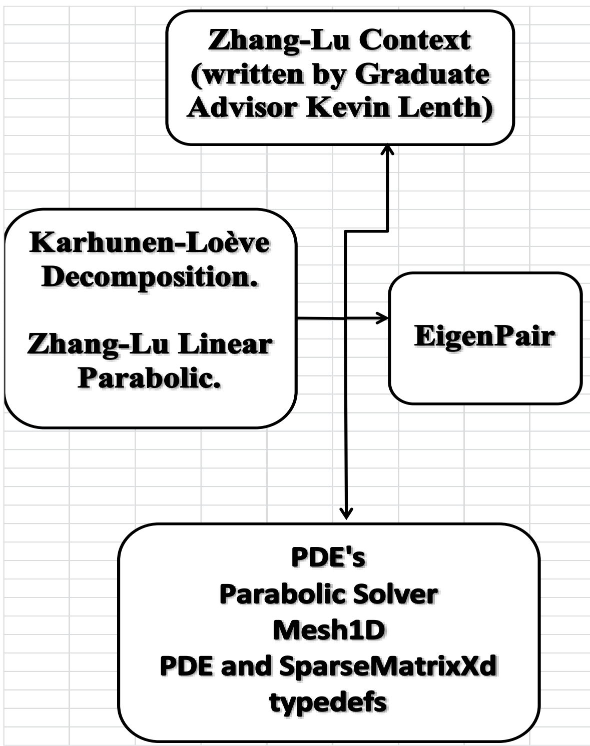


Figure 1. Code Interdependencies.

- Figure 1 shows the mutual dependence of different parts of the code.
- The base element for solving these problems is the class Mesh1D, which constructs the mesh and computes the stiffness matrix and load vector.
- The class ParabolicSolver1D solves linear deterministic parabolic equations.

Code Description

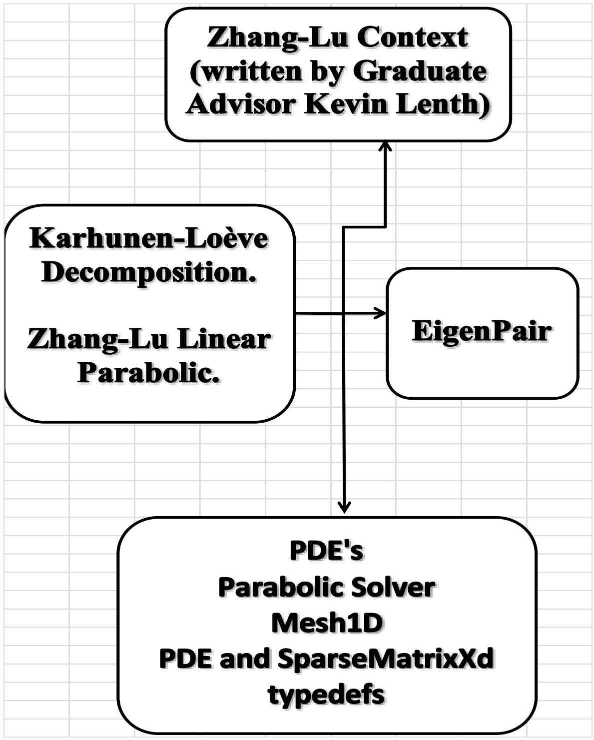


Figure 1. Code Interdependencies.

- Several classes are necessary in application of the Zhang-Lu method.
- KarhunenLoeve computes eigenfunctions and creates a class, EigenPair, to sort them conveniently.
- The final class, ZhangLuContextLinearParabolic1D, is hooked into Kevin Lenth's preexisting framework. This will compute, among other things, variance and expectation so the results can be compared to Monte Carlo trials.

Varying Sigma (σ)

- Let the disparity be the difference between the expectations and variances for Monte Carlo and Zhang-Lu solutions.
- Let us examine the dependence of the disparity on the value of sigma. This is calculated in the following way.

$$\frac{||ZhangLu\ solution - Monte\ Carlo\ solution||}{||Monte\ Carlo\ solution||}$$

- where $||solution||$ means the L^2 norm of the solution.

Varying Sigma (σ)

Figure 2. Disparity in Expectation

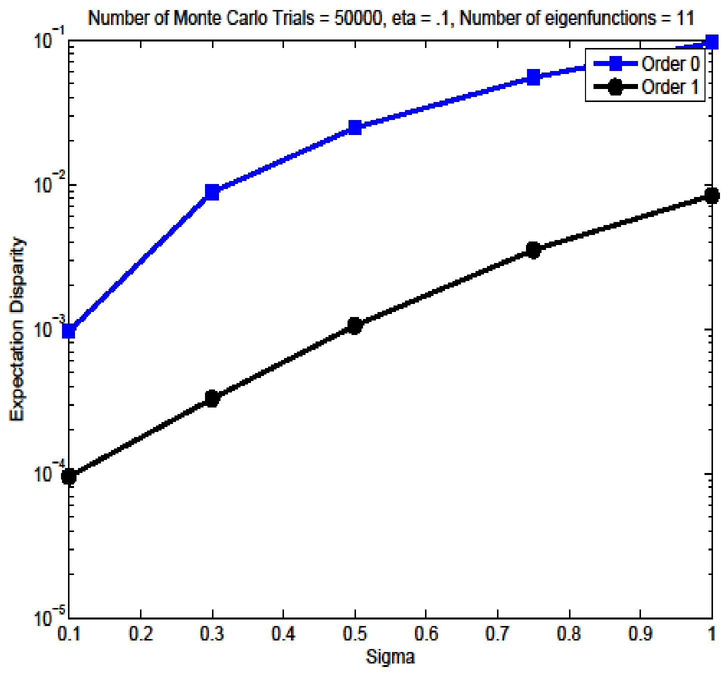


Figure 2. Disparity in Expectation.

Figure 3. Disparity in Variance

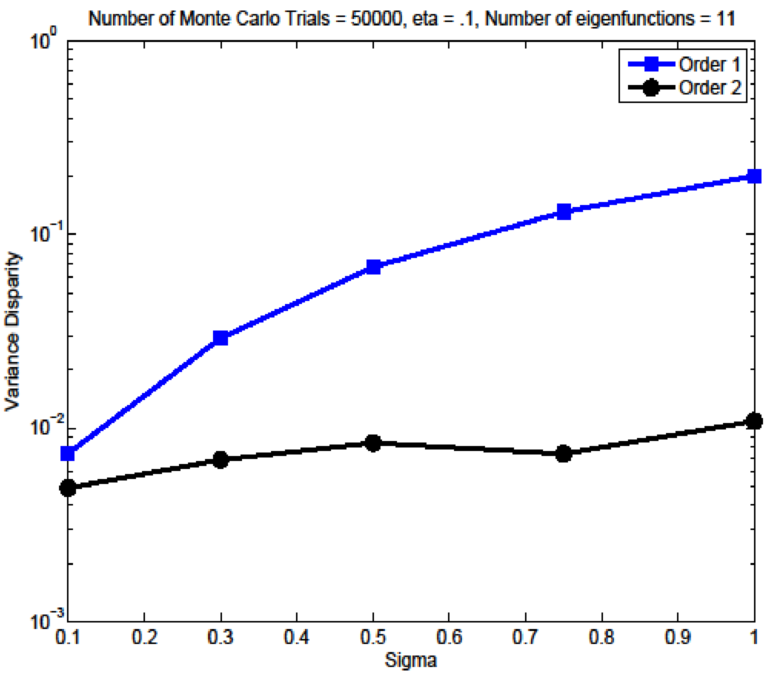


Figure 3. Disparity in Variance.

Varying Sigma (σ)

- As expected, it is shown in Figure 2 that as standard deviation (σ) increases, the disparity between the two solutions also increases.
- A higher value of σ corresponds to a higher level of "randomness".
- It is also observed that disparity for Order 1 is lower than in Order 0 as expected. As more terms are included, accuracy increases.

Varying Sigma (σ)

- Similarly to Figure 2, the variance also increases as σ increases.
- Note that the disparity is again lower in the higher order since more terms are included.

Varying K (number of terms in Karhunen-Loève expansion)

Figure 4. Disparity in Expectation for Coarse Mesh

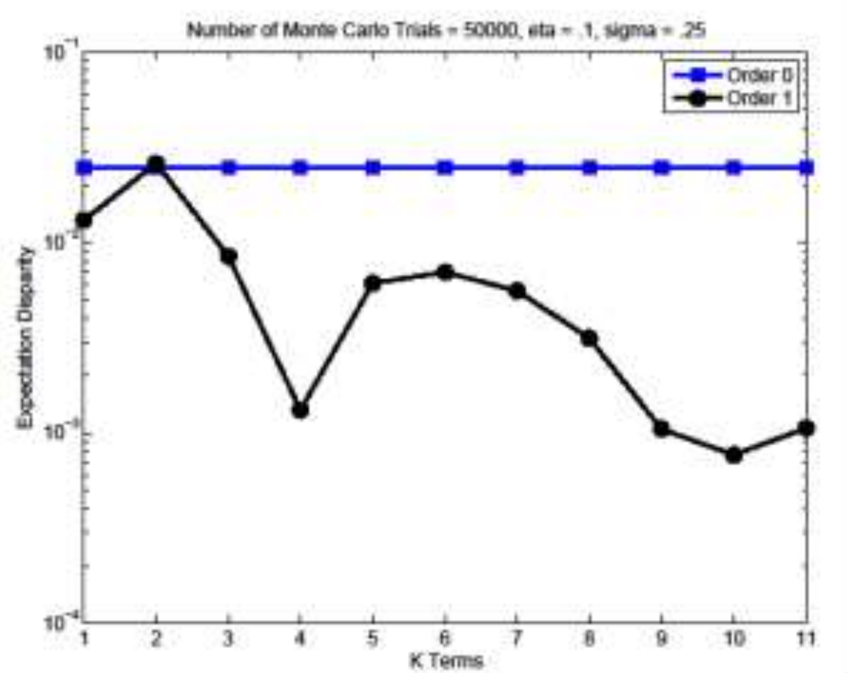
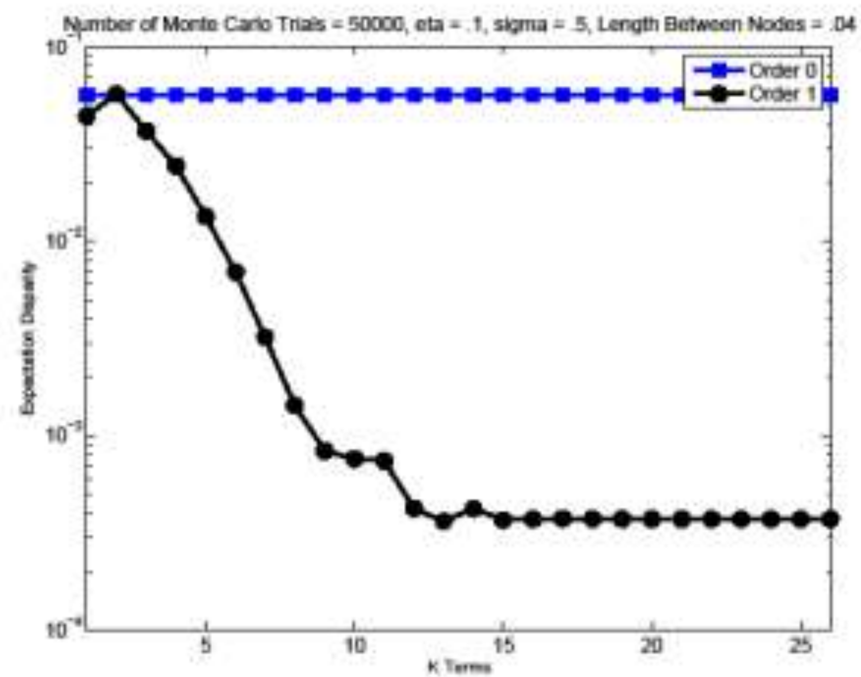


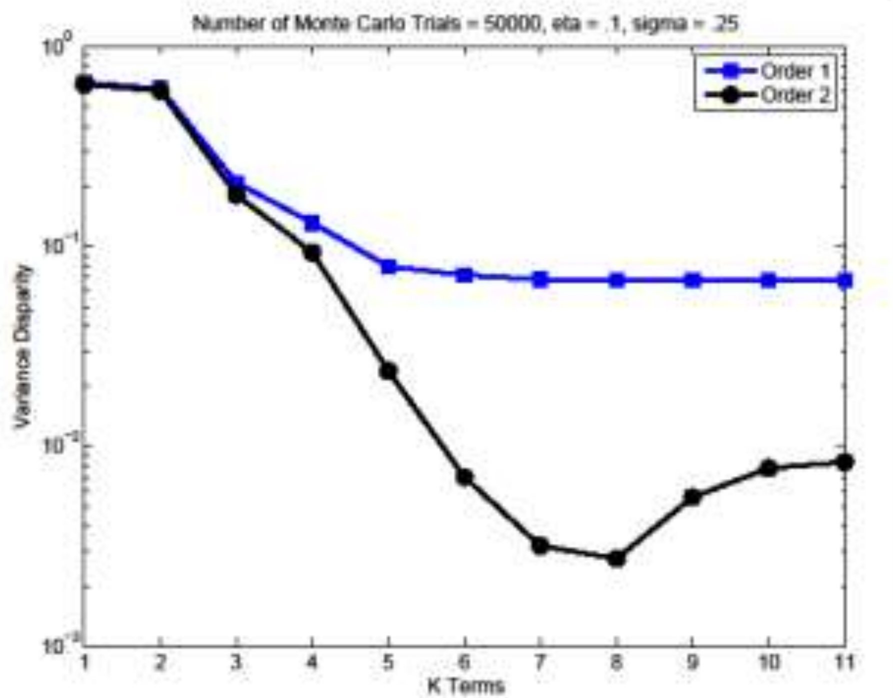
Figure 5. Disparity in Expectation for Fine Mesh

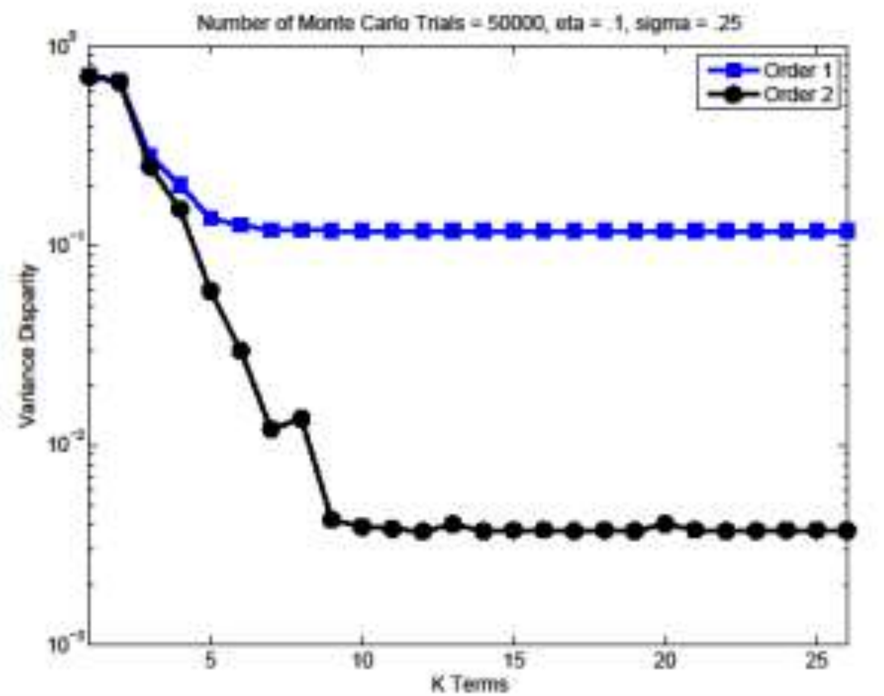


Varying K (number of terms in Karhunen-Loève expansion)

- As can be seen by juxtaposing Figures 4 and 5, it appears that the results improve and disparity decreases as the mesh becomes more refined.
- Also note that Figure 5 his almost always decreasing, whereas Figure 4 has a sharp increase in disparity.
- Note expectation for Order 0 does not change as the number of K terms changes since it is deterministic and therefore not affected by Karhunen-Loève expansion terms

Varying K (number of terms in Karhunen-Loève expansion)





Varying K (number of terms in Karhunen-Loève expansion)

- Similarly, it can be seen that these results again appear to improve as the mesh gets finer.
- Note that in Figure 6, variance begins to increase, whereas in Figure 7 it is seen that it remains low.
- All of this reaffirms the assertion by Zhang and Lu that the solution will improve as more terms are used in the solution.

CONCLUSION

- In the case of this project, it can be inferred that as standard deviation increases, the amount of disparity also seems to increase in the expectation as well as the variance.
- It can also be inferred that using a coarse mesh will not yield as accurate of results for expectation and disparity as a fine mesh using the same functions and covariance.

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QUESTIONS?