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A SIMPLE PROOF OF FIEDLER'S CONJECTURE CONCERNING ORTHOGONAL MATRICES

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ABSTRACT. We give a simple proof that an $n \times n$ orthogonal matrix with $n \geq 2$ which cannot be written as a direct sum has at least $4n - 4$ nonzero entries.

1. The result. What is the least number of nonzero entries in a real orthogonal matrix of order n? Since the identity matrix I_n is orthogonal the answer is clearly n. A more interesting question is: what is the least number of nonzero entries in a real orthogonal matrix which, no matter how its rows and columns are permuted, cannot be written as a direct sum of (orthogonal) matrices? Examples of orthogonal matrices of each order $n \geq 2$ which cannot be written as a direct sum and which have $4n - 4$ nonzero entries are given in [**1**]. M. Fiedler conjectured that an orthogonal matrix of order $n \geq 2$ which cannot be written as a direct sum has at least $4n - 4$ nonzero entries.

Using a combinatorial property of orthogonal matrices, Fiedler's conjecture was proven in [**1**]. A $(0, 1)$-matrix A of order n is *combinatorially orthogonal* provided no pair of rows of A has inner product 1 and no pair of columns of A has inner product 1. Clearly, if Q is an orthogonal matrix of order n, then the $(0, 1)$-matrix obtained from Q by replacing each of its nonzero entries by a 1 is combinatorially orthogonal. A quite lengthy and complex combinatorial argument is used in [**1**] to show that if A is a combinatorially orthogonal matrix of order $n \geq 2$ and A cannot be written as a direct sum, then A has at least $4n - 4$ nonzero entries. Clearly this result implies Fiedler's conjecture. In this note we give a simple matrix theoretic proof of Fiedler's conjecture.

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Theorem 1.1. *Let Q be an orthogonal matrix of order n of the form*

$$Q = \begin{bmatrix} U & O \\ V & W \end{bmatrix}$$

where U is a $k \times k+l$ matrix, and W is an $m+l \times m$ matrix for some positive integers k and m and nonnegative integer l with $k+l+m=n$. Then the rank, $r(V)$, of V equals l.

Proof. Since the rows of U are linearly independent $r(U) = k$. Similarly, $r(W) = m$. Since the rank of a sum of matrices is less than or equal to the sum of the ranks of the matrices,

$$r(Q) \leq r(U) + r(W) + r(V).$$

Thus $l \leq r(V)$, since $r(Q) = k + l + m$. Because Q is orthogonal, the rows of V belong to the orthogonal complement in R^{k+l} of the space spanned by the rows of U. Since $r(U) = k$, this implies that $r(V) \leq l$. Therefore $r(V) = l$. □

We note that, by taking $l = 0$ in Theorem 1.2, we have $V = O$; and hence an orthogonal matrix Q of order n can be written as a direct sum of matrices (after possibly permuting its rows and columns) if and only if Q contains a zero submatrix whose dimensions sum to n.

Corollary 1.2. *Let*

$$Q = \begin{bmatrix} U & O \\ V & W \end{bmatrix}$$

be an $n \times n$ orthogonal matrix where U is $k \times k+1$ and W is $l+1 \times l$, $k+l = n-1$ and $k, l \geq 1$. Then there exist nonzero vectors x and y such that $V = xy^T$, and both

$$U' = \begin{bmatrix} U \\ y^T \end{bmatrix} \quad \text{and} \quad W' = \begin{bmatrix} x & W \end{bmatrix} \tag{1}$$

are orthogonal matrices.

Proof. By Theorem 1.1, V has rank one. Hence, there exist vectors x and y such that $V = xy^T$. Since Q is orthogonal, the sum of the squares of the entries in its first k rows equals k, and the sum of the squares of the entries in its first $k+1$ columns equals $k+1$. Hence, the sum of the squares of the entries in V equals 1. It follows that $(x^Tx)(y^Ty) = 1$. Thus, by replacing x by $(1/\sqrt{x^Tx})x$ and y by $\sqrt{x^Tx}y$, we may assume that $x^Tx = 1$ and $y^Ty = 1$. Since Q is orthogonal, y^T is orthogonal to each row of U, and x is orthogonal to each column of W. The corollary now follows. □

We now prove Fiedler's conjecture. We let $\#(A)$ denote the number of nonzero entries in the matrix A.

Theorem 1.3. *Let Q be an orthogonal matrix of order $n \geq 2$ which cannot be written as a direct sum of matrices (no matter how its rows and columns are permuted). Then Q has at least $4n-4$ nonzero entries.*

Proof. The proof is by induction on n. First suppose that Q contains a $k\times l$ zero submatrix for some positive integers k and l with $k+l = n-1$. Without loss of generality we may assume that

$$Q = \begin{bmatrix} U & O \\ V & W \end{bmatrix}$$

where U is $k \times k+1$, and W is $l+1 \times l$. By Corollary 1.2, there exist x and y such that $V = xy^T$ and the matrices U' and W' in (1) are orthogonal matrices.

Suppose that U' can be written as a direct sum of two matrices. Then U' contains an $r \times s$ zero submatrix which does not intersect the last row of U' for some positive integers r and s with $r + s = k + 1$. It follows that Q contains an $r \times s + (n - k - 1)$ zero submatrix. Hence, by the observation immediately after Theorem 1.1, Q can be written as a direct sum of matrices. This contradicts our assumptions. Thus, U' cannot be written as a direct sum of matrices. A similar argument shows that W' cannot be written as a direct sum of matrices.

Clearly,

$$\#(Q) = \#(U') + \#(W') - 1 + (\#(y) - 1)(\#(x) - 1).$$

By induction U' has at least k nonzero entries, and W' has at least $4l$ nonzero entries. Thus,

$$\begin{aligned}\#(Q) &\geq 4k + 4l - 1 + (\#(y) - 1)(\#(x) - 1)\\ &= (4n - 4) - 1 + (\#(y) - 1)(\#(x) - 1).\end{aligned}$$

Since Q has no $r \times s$ zero submatrix with $r + s \geq n$, $\#(y) \geq 2$ and $\#(x) \geq 2$. Therefore, $\#(Q) \geq 4n - 4$.

Now suppose that Q does not contain a $k \times l$ zero submatrix for any positive integers k and l with $k + l = n - 1$. If $n = 2$, then each entry of Q is nonzero and hence $\#(Q) \geq 4(n - 1)$. Assume that $n \geq 3$. Then each row and column of Q has at least 3 nonzero entries. Thus, if $n = 3$, then $\#(Q) > 4(n - 1)$.

Assume that $n \geq 4$. If each row and column of Q has at least 4 nonzero entries, then $\#(Q) \geq 4n > 4(n - 1)$. Suppose that some row or column of Q has exactly 3 nonzero entries. We may assume without loss of generality that row 1 of Q has exactly 3 nonzero entries, and that these occur in columns 1, 2 and 3. Let

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} & 0 & \cdots & 0 \\ u & v & w & & X & \end{bmatrix},$$

where X is $n - 1 \times n - 3$.

By Theorem 1.1, the rank of $[u\,v\,w]$ is 2. Without loss of generality we may assume that u and v are linearly independent.

Since each of u, v and w is orthogonal to each column of X,

$$Q' = [\, u' \quad v' \quad X \,]$$

is an orthogonal matrix of order $n - 1$, where u' and v' are the vectors obtained from u and v by applying the Gram-Schmidt process.

Suppose that Q' can be written as a direct sum of two matrices. Then there exist positive integers r and s with $r + s \geq n - 2$ such that X contains an $r \times s$ zero submatrix. It follows that Q contains an $r + 1 \times s$ zero submatrix, which contradicts our assumptions. Hence Q' cannot be written as a direct sum of matrices.

By the induction hypothesis, $\#(Q') \geq 4n - 8$. Clearly $\#(u') = \#(u)$, and

$$\#(Q) = \#(Q') - \#(v') + 3 + \#(v) + \#(w).$$

Thus it follows that

$$\#(Q) \geq 4n - 5 + \#(v) + \#(w) - \#(v').$$

Since rows $2, 3, \ldots, n$ of Q are orthogonal to the first row of Q, no row of $[u\,v\,w]$ contains exactly one nonzero entry. Thus, each row of $[v\,w]$ contains at least as many nonzero entries as the corresponding row of v'. Since the second and third columns of Q are orthogonal, some row of $[v\,w]$ has no zero entries. Thus, for some i, row i of $[v\,w]$ has more nonzero entries than row i of v'. It follows that $\#(Q) \geq 4n - 4$. □

The techniques used in the proof of Theorem 1.3 can be used to classify, as was done in [**1**], the orthogonal matrices of order n which cannot be written as a direct sum and which have exactly $4n-4$ nonzero entries.

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