Turán Numbers of Vertex-disjoint Cliques in r-partite Graphs
Final Honors Project, University of Wyoming

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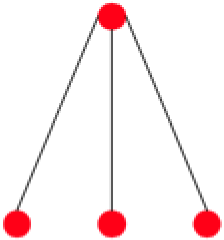
History and Motivation

- Graph theory is the math of connections
- Applications in other fields, both abstract and applied
- Historically began with Euler: "The Seven Bridges of Königsberg" (1736)
- Erdős is considered the father of the field (early 20th century)

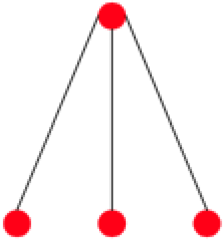
Definitions

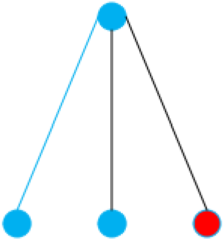
- Graph: A **graph** G is a pair of sets $G = (V, E)$, where V is a fixed set of vertices, and the edge set E is a set of pairs of distinct elements from V. We often write V as $V(G)$ and E as $E(G)$.



Definitions

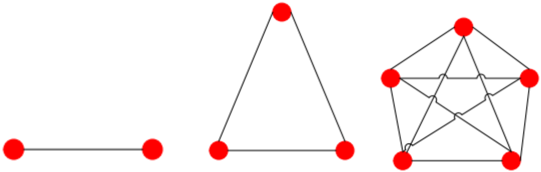
- Graph
- Subgraph: A **subgraph** H of G is a pair of sets $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$, which is itself a graph. If H is a subgraph of G, we write $H \subseteq G$.





Definitions

- Graph
- Subgraph
- Clique: A **clique** is a graph or subgraph in which every vertex is adjacent to every other vertex. A clique of size r is a **complete** graph on r vertices.



Definitions

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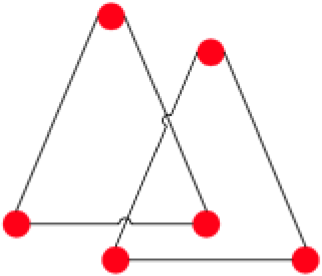
  More than one cliques present that do not share vertices are called **vertex-disjoint cliques**.



Definitions

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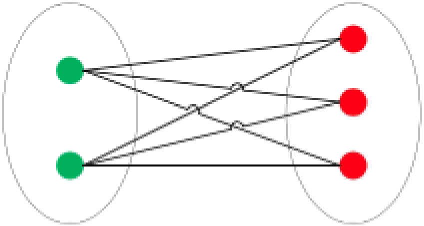
- r-Partite: A graph G is called **r-partite** if there are r partitions of the vertex set $V(G) = V_1 \cup V_2 \cup ... \cup V_r$ such that if y and y are both in the same V_i, then $xy \notin E(G)$.
  r-Partite graphs can also be **complete**.



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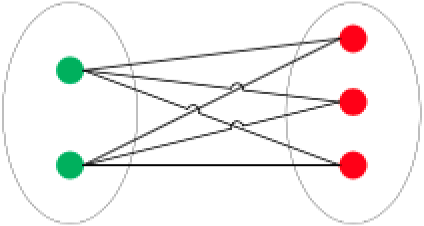

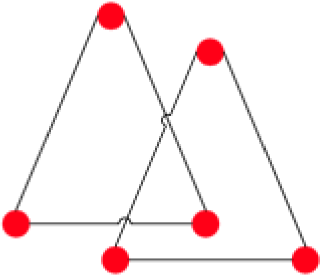

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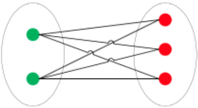


How can we count edges?

Notation

- A complete graph is often denoted with a K.
- k copies of K is denoted kK
- A complete r-partite graph is denoted $K_{n_1,\ldots n_r}$, where there $n_1, \ldots n_r$ are the number of vertices in each part
- We denote k vertex-disjoint cliques of size r as kK_r

Turán Numbers

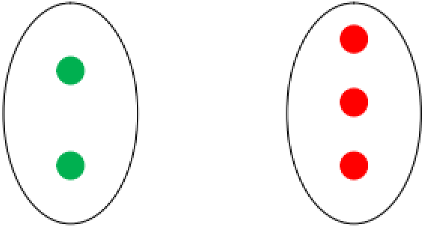
ex(G,H)

What is the maximum number of edges which a subgraph of G may have and still contain no copy of H?

Turán Numbers

What is the maximum number of edges which a subgraph of G may have and still contain no copy of H?
Let $G = K_{2,3}$ and $H = 2K_2$
What is $ex(K_{2,3}, 2K_2)$?

Turán Numbers



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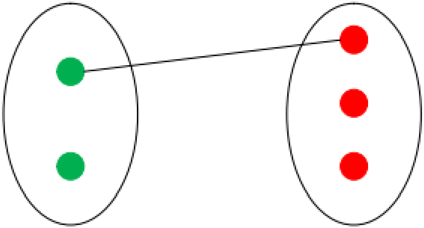

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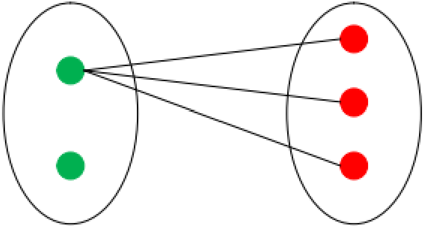

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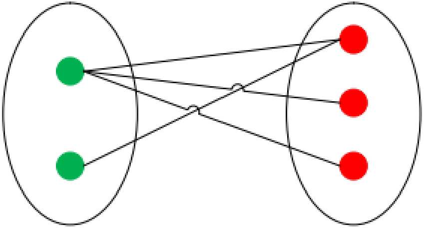

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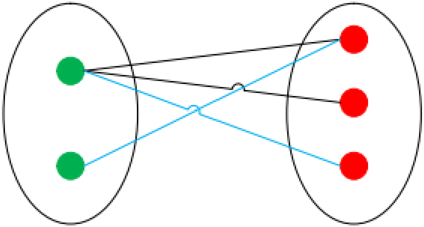

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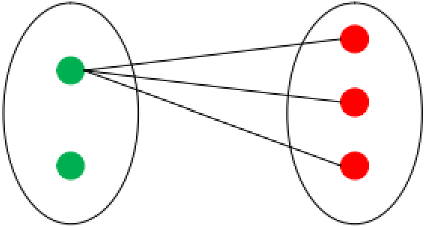
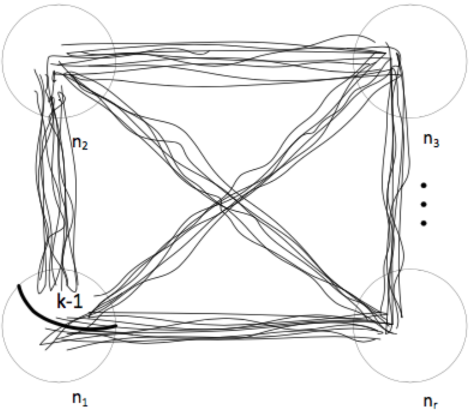


Therefore, $ex(K_{2,3}, 2K_2) = 3$

Our Theorem

For any integers $1 \leq k \leq n_1 \leq ... \leq n_r$, we have

$$ex(K_{n_1,n_2,...,n_r}, kK_r) = \sum_{1 \leq i < j \leq r} n_i n_j - n_1 n_2 + n_2(k-1)$$



Proof Ideas

- First half of the proof is the lower bound
- Second half of the proof is the upper bound
  - Proof strategy - **induction**
  - Double induction on $n_1 + k$
  - Two lemmas as base cases where we are only looking for one clique, and where each part size is equal

Results

- The extremal number is *at least* the number we claim (lower bound)
- The extremal number is *at most* the number we claim (upper bound)
- Therefore, the extremal number is *exactly* the number we claim
- Characteristics of host and forbidden graph allow us to have equality

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Questions?