A Binary Exploration

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What to Expect

- I will:

1. Explain the binary number system

    How to:

    -Generate binary from a number

    -Add binary

2. Explain the base-b number system
3. Explain the double-base number system

The Binary Number System

- The modern binary number system was discovered by a mathematician named Gottfried Leibniz and was introduced in his paper *Explication de l'Arithmétique Binaire* in 1703.
- Other people played with the idea, but Leibniz was the first to publish in the form we use today.

The Binary Number System cont…

- The binary number system is a numerical system that uses only zeros and ones and has base 2.
- Think of your fingers as base 10 representation and if you only had 2, then they would represent base 2.
- See computer applications, power on, power off. (Continuous flow of power 1010, power on, power off, power on, power off).
- Converting numbers to binary is remarkably easy once you get the hang of it.
- First you pick a number, let's say 137.
- Then recall that the binary system is base two, so you think about what number base two is closest to 137:

  -i.e. $2^0=1$, $2^1=2$, $2^2=4$, $2^3=8$, $2^4=16$, $2^5=32$, $2^6=64$, $2^7=128$, $2^8=256$.
- We can see the closest number is $2^7=128$.

The Binary Number System cont…

- Next we take the number we just found (128) and subtract it from our original number:

  137-128=9
- Then we begin the process all over again for the number 9.
- The closest number base 2 to 9 is 2^3=8, so we subtract that from 9 and get 1.
- And if you will recall, 2^0=1, so now we can put these numbers together and find our binary representation.

The Binary Number System cont…

- Finally, we group all the base 2 numbers that we found to generate 137:

  $2^7+2^3+2^0=137$

- But, this will not give us the binary representation, so we need to fill in the gaps and insert some coefficients:

  $1*(2^7)+0*(2^6)+0*(2^5)+0*(2^4)+1*(2^3)+0*(2^2)+0*(2^1)+1*(2^0)$
  $=137$

- We can now write the number 137 in binary, all we have to do is put the coefficients of the numbers together: 10001001.

The Binary Number System cont…

- Another less involved example is 13.

  13-8=5, 5-4=1, 1-1=0, so the binary representation for $13=1*(2^3)+1*(2^2)+0*(2^1)+1*(2^0)=1101$.

After you understand how to convert numbers into binary, you can then begin to add, subtract, multiply and divide (but we will focus on adding).

Adding Binary Numbers

- Adding binary numbers is relatively simple, you just have to remember the binary rule that 1+1=10.
- So, say you want to add 13+7. (We can find that 13+7=20 and 13 converted into binary is 1101, 7 is 111, and 20 is 10100.)
- Now adding 1101

  + 111

  10100
- We started from the right side, added 1+1=10 and carried the 1, then 0+1=1, but recall we have a carried 1, so 0+1+1=10 and carry another 1, then 1+1=10, but we carried a 1 again, so we leave a 1 behind, and carry the final 1 and 1+1=10.

Adding Binary Numbers cont…

- We can see check our binary addition by checking our answer against the correct answer converted into binary.
- When we do this, we see that 10100=10100, so we know the arithmetic is correct.
- Subtraction and multiplication follow a similar algorithm.

Base B-Representation

- The binary number system is not the only system that we can use to convert numbers.
- We can also use base b-representation which is just a way of denoting that you can have a number system using any positive number except 1.
- So, if I was using a base 7 number system, the representation of 7013 base 7 is:

  $2*7^4+6*7^3+3*7^2+0*7^1+6*7^0=26306$

- And if we were using a base 10 (the base we usually use), the representation of 7013 base 10 is:

  $7*10^3+0*10^2+1*10^1+3*10^0=7013$

Final Thoughts on Binary and Base B

- The characteristic that both binary and base b representation can boast is uniqueness.
- Meaning, there is only one way to represent any number in binary or base b.
- This is important because if there is only one way to represent a number, there will be no confusion about what number to use when building code or doing any calculations base 10. 724 is the only way to represent the number seven hundred twenty four, you cannot represent that number by changing a digit because it won't be the same number.

Double Base Numerical System

- The double base number system is a system that uses two values to represent a number rather than one.

  -Meaning that instead of just using 2^n in binary

  we are moving to a system that uses 2^i3^j, where i,j=all positive intigers numbers.

Double Base Numerical System cont…

- The tricky aspect of double base arises when figuring out how to represent different numbers. We are no longer working with one number raised to a power, we now have to worry about two numbers raised to a power and then multiplied together.
- A number like 1046 can be represented as:

  $2^1 3^0 + 2^2 3^1 + 2^3 3^1 + 2^4 3^2 + 2^5 3^3$

  But, it can also be represented as:

  $2^1 3^0 + 2^2 3^3 + 2^4 3^2 + 2^5 3^3$

Double Base Numerical System cont…

- I began my research into double base finding the representation for smaller numbers (i.e. 1-1000) and doing this with just a calculator and my knowledge of powers and multiplication was not a problem.
- But, when looking at numbers that were greater than 1000, just using a calculator proved to be very cumbersome and time consuming.
- The other problem with that approach is that it does not guarantee a unique representation.
- So, my research advisor pushed me to understand how I was finding the representation for the smaller numbers.

Double Base Numerical System cont…

- I was using the same principles behind finding binary representation, simply picking the largest number base 2^i3^j subtracting it from the number I had chosen and then repeating that step until I reached zero.
- In the article "Theory and Applications of the Double-Base Number System," Vassil S. Dimitrov, Graham A. Jullien, and William C. Miller call this method a Greedy Algorithm.
- This algorithm comes in very handy when finding the representation for large numbers.

The Greedy Algorithm

- The definition of an algorithm (from wordnetweb) is a precise rule (or set of rules) specifying how to solve some problem.
- So, a Greedy Algorithm is a precise set of rules that outline how to find the double base representation for numbers.
- Step 1: Pick any positive integer n and find the largest 2^i3^j that is smaller than your number and subtract them : $n - 2^i3^j = n_2$.
- Step 2…x: Repeat step 1 with your n_2 value and continue doing so until your final number =0.

The Greedy Algorithm cont…

- This does not sound too difficult or time consuming, in theory, but unless you have another tool at your disposal, it proves to be very tedious.
- This other tool is a table outlining what each value for 2^i3^j equals up to i,j=7.
- For larger calculations than I will show you a table with larger i,j values is required, but for the purposes of this presentation, I will use a smaller table.

The Greedy Algorithm cont…

$2^0 3^0$	$2^1 3^0$	$2^2 3^0$	$2^3 3^0$	$2^4 3^0$	$2^5 3^0$	$2^6 3^0$	$2^7 3^0$
$2^0 3^1$	$2^1 3^1$	$2^2 3^1$	$2^3 3^1$	$2^4 3^1$	$2^5 3^1$	$2^6 3^1$	$2^7 3^1$
$2^0 3^2$	$2^1 3^2$	$2^2 3^2$	$2^3 3^2$	$2^4 3^2$	$2^5 3^2$	$2^6 3^2$	$2^7 3^2$
$2^0 3^3$	$2^1 3^3$	$2^2 3^3$	$2^3 3^3$	$2^4 3^3$	$2^5 3^3$	$2^6 3^3$	$2^7 3^3$
$2^0 3^4$	$2^1 3^4$	$2^2 3^4$	$2^3 3^4$	$2^4 3^4$	$2^5 3^4$	$2^6 3^4$	$2^7 3^4$
$2^0 3^5$	$2^1 3^5$	$2^2 3^5$	$2^3 3^5$	$2^4 3^5$	$2^5 3^5$	$2^6 3^5$	$2^7 3^5$
$2^0 3^6$	$2^1 3^6$	$2^2 3^6$	$2^3 3^6$	$2^4 3^6$	$2^5 3^6$	$2^6 3^6$	$2^7 3^6$
$2^0 3^7$	$2^1 3^7$	$2^2 3^7$	$2^3 3^7$	$2^4 3^7$	$2^5 3^7$	$2^6 3^7$	$2^7 3^7$

The Greedy Algorithm cont…

1	2	4	8	16	32	64	128
3	6	12	24	48	96	192	384
9	18	36	72	144	288	576	1152
27	54	108	216	432	864	1728	3456
81	162	324	684	1296	2592	5184	10368
243	486	972	1944	3888	7776	15552	31104
729	1458	2916	5832	11664	23328	46656	93312
2182	4374	8748	17496	34992	69984	139968	279936

The Greedy Algorithm cont…

- Now, we can implement the Greedy Algorithm on a large number such as 169,213.
- First we look at the table and find that 139,968 (which is $2^6 3^7$) is the largest number in the table that is smaller than 169,213, so we take 169213-139968=29245.
- Next we take 29245 and do the same thing and find 23328 ($2^5 3^6$), so 29245-23328=5917.
- Follow the same procedure for 5917, and find 5184 ($2^6 3^4$), so 5917-5184=733.
- Once again for 733 we find 729 ($2^0 3^6$), so 733-729=4.
- Finally, do the same for 4 and find 4 ($2^2 3^0$), so 4-4=0.

The Greedy Algorithm cont…

- Then, sum all of the 2^i3^j values and it should equal the number you began with:

  $2^6 3^7 + 2^5 3^6 + 2^6 3^4 + 2^0 3^6 + 2^2 3^0 = 169{,}213.$

- As you can see this representation takes only five terms, whereas if you were to represent this number in binary it would take 18. (101001010011111101)
- The great thing about this algorithm is that it allows one to find a unique representation for any number, whereas without it I could not do that.

The Greedy Algorithm cont…

- Along with uniqueness, the algorithm also has some other interesting characteristics.
- For instance, when I started using the greedy algorithm, I thought there would only be one “good” (the way that uses fewest terms) way to represent a number, but it turns out that is not the case.
- I found a total of 2 ways to represent 169213 using 5 terms.
- This is interesting because it shows that even when I try to choose other ways to represent a number, I can still find a way that can be represented using few terms.

The Greedy Algorithm cont…

- This shows that this algorithm is a strong one because it doesn't lend itself to a lot of terms.
- Of course, if I wanted to begin experimenting with ways that I could generate a large number of terms (which I did), I could do that by simply closing my eyes and pointing at a number subtracting it from my original number and starting over again. But, even this "closed eye" method can generate some representations that have fewer terms than the binary.

Column and Row Reduction

- If we refer back to the table of 2^i3^j values, there is a simple calculation we can do on special combinations of numbers. Either numbers that are right next to each other in a row or numbers that are right next to each other in a column.

$2^0 3^0$	$2^0 3^1$	$2^0 3^2$	$2^0 3^3$
$2^1 3^0$	$2^1 3^1$	$2^1 3^2$	$2^1 3^3$
$2^2 3^0$	$2^2 3^1$	$2^2 3^2$	$2^2 3^3$
$2^3 3^0$	$2^3 3^1$	$2^3 3^2$	$2^3 3^3$

Column and Row Reduction cont…

- Now choose two numbers that are beside each other in a column, like $2^1 3^1$ and $2^2 3^1$ in order to reduce these numbers into something smaller, we must factor out a common term: $2^1 3^1 + 2^2 3^1 = 2^1 3^1(1+2) = 2^1 3^1(3) = 2^1 3^2$,so we move visually from:

$2^0 3^0$	$2^0 3^1$	$2^0 3^2$	$2^0 3^3$
$2^1 3^0$	$2^1 3^1$	$2^1 3^2$	$2^1 3^3$
$2^2 3^0$	$2^2 3^1$	$2^2 3^2$	$2^2 3^3$
$2^3 3^0$	$2^3 3^1$	$2^3 3^2$	$2^3 3^3$

=>

$2^0 3^0$	$2^0 3^1$	$2^0 3^2$	$2^0 3^3$
$2^1 3^0$	$2^1 3^1$	$2^1 3^2$	$2^1 3^3$
$2^2 3^0$	$2^2 3^1$	$2^2 3^2$	$2^2 3^3$
$2^3 3^0$	$2^3 3^1$	$2^3 3^2$	$2^3 3^3$

Column and Row Reduction cont…

- Row reduction is calculated using the same idea behind column reduction you are simply reducing the rows rather than the columns.
- This idea of row reduction comes in handy if you have a number in which you have numbers from the same row or column because by doing the reduction you can reduce the number of terms used to represent the number.

Final Thoughts

- If you take nothing else away from this presentation please remember that binary and double base representation, found using the greedy algorithm, is a unique representation.

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Works Cited

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Multiplying Binary Numbers

- In order to multiply binary numbers, you follow the same rules you do when multiplying in the "normal" number system (base 10), but once you get to the addition, you follow the procedure I illustrated previously.
- As an example we can multiply 13x7=91 (91 in binary is 1011011).

Multiplying Binary Numbers cont…

- So, 1101

  x 111

  1101

  1101

  + 1101

  1011011
- And if we check, 91=1011011=1011011, so the arithmetic is correct.

Subtracting Binary Numbers

- The idea behind subtracting binary numbers is very similar to subtracting numbers base 10.